Computing Gorenstein colength of Artin rings

Roser Homs

Universitat de Barcelona

Joint work with J. Elias

BASIC SETUP:

$\mathbf{k}$ arbitrary field,
$R = \mathbf{k}[\![x_1, \dots, x_n]\!]$ ring of formal power series of dimension n,
$\mathbf{m} = (x_1 \dots, x_n)$ unique maximal ideal,
$\mathbf{k} = R/\mathbf{m}$ residue field.

Theorem (Cohen structure theorems)

Let $(A, \mathbf{n}, \mathbf{k})$ be an equicharacteristic Artin local ring, then

$$A \cong \frac{\mathbf{k}[\![x_1, \dots, x_n]\!]}{I}$$

for some $\mathbf{m}$-primary ideal I of $\mathbf{k}[\![x_1, \dots, x_n]\!]$.

Definition

An Artin ring $A = R/I$ which satisfies any of the equivalent conditions below is called a **Gorenstein ring** of dimension zero:

1. $\mathrm{id}_A(A) < \infty$.
2. A is injective as a module over itself.
3. $A \cong E_A(\mathbf{k}) \cong \omega_A$.
4. $\tau(A) := \dim_{\mathbf{k}} \mathrm{soc}_A(A) = 1$, where $\mathrm{soc}_A(A) := (0 :_A \mathbf{n})$.
5. The ideal (0) in A is irreducible.

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Gorenstein	$$\frac{\mathbf{k}[\![x_1, x_2, x_3]\!]}{(x_1x_2, x_1x_3, x_2x_3, x_2^2 - x_1^3, x_3^2 - x_1^3)}$$

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C.-M.	$$\frac{\mathbf{k}[\![x_1,x_2]\!]}{(x_1^2,x_1x_2,x_2^4)}$$ $$\frac{\mathbf{k}[\![x_1,x_2]\!]}{(x_1^2,x_1x_2^2,x_2^4)}$$

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Definition (Ananthnarayan 08')

The **Gorenstein colength** of an Artin local ring A is

$$gcl(A) = \min\{\ell(G) - \ell(A) \mid G \twoheadrightarrow A, G \text{ local Artin Gorenstein}\}$$

We call any G reaching this minimum **minimal Gorenstein cover** of A.

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Known bounds (Ananthnarayan 08'):

$$\ell\left(A/\omega^*(\omega)\right) \leq \min\{\ell(A/\mathfrak{q}) : \mathfrak{q} \cong \mathfrak{q}^+\} \leq \mathrm{gcl}(A) \leq \ell(A)$$

Notation:
ω_A canonical module of A,
$\omega_A^*(\omega_A) = \langle f(\omega_A) : f \in \mathrm{Hom}_A(\omega_A, A)\rangle$ trace ideal of ω_A,
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Theorem (Ananthnarayan 08')

Let $A = R/I$, $I \subseteq \mathbf{m}^6$ and assume $2 \in A^*$. TFAE:

1. $\mathrm{gcl}(A) \leq 2$,
2. *There exist an ideal $\mathfrak{q} \in A$ with $\ell(A/\mathfrak{q}) \leq 2$ such that $\mathfrak{q} \cong \mathfrak{q}^+$.*

Macaulay duality provides an order-reversing bijection between Artin local rings $A = R/I$ and a sub-R-module of the ring of polynomials:

$$\{\mathbf{m}-\text{primary ideals of } \mathbf{k}[\![x_1,\dots,x_n]\!]\} \longleftrightarrow \{\text{f.g. sub-}\mathbf{k}[\![x_1,\dots,x_n]\!]\text{-modules of } \mathbf{k}[y_1,\dots,y_n]\}$$

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Definition

S is an R-module with the **CONTRACTION** structure:

$$\begin{array}{ccc} R \times S & \longrightarrow & S \\ (x_1^{\alpha_1}\cdots x_n^{\alpha_n}, y_1^{\beta_1}\cdots y_n^{\beta_n}) & \mapsto & x^\alpha \circ y^\beta = \begin{cases} y_1^{\beta_1-\alpha_1}\cdots y_n^{\beta_n-\alpha_n}, & \beta_i \geq \alpha_i; \\ 0, & \textit{otherwise}. \end{cases} \end{array}$$

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$$I^\perp = \langle \mathbf{y}_2^3, \mathbf{y}_1\mathbf{y}_2 \rangle$$

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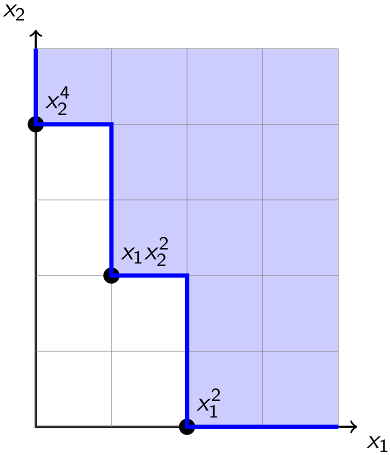

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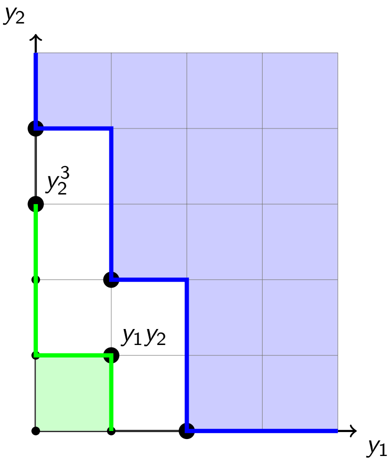

$I^\perp = \langle F_1, \dots, F_n \rangle$, $S_{\leq i} = \{F \in S \mid \deg(F) \leq i\}$ sub-R-module of S.

$$(I^\perp)_i = \frac{I^\perp \cap S_{\leq i} + S_{<i}}{S_{<i}}.$$

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Hilbert function of an Artin local ring $A = R/I$ of socle degree s:

$$H_A(i) = \begin{cases} 1, & \text{if } i = 0, \\ n, & \text{if } i = 1, \\ \dim_{\mathbf{k}} (I^\perp)_i, & \text{if } 2 \leq i \leq s, \\ 0, & \text{if } i \geq s+1. \end{cases}$$

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Socle degree: $\operatorname{soc}\deg(A) = \max\{\deg(F_1), \ldots, \deg(F_n)\}$.
Cohen-Macaulay type: $\tau(A) = \mu_R(I^\perp)$.

Proposition (Characterization of Artin Gorenstein rings)

An Artin local ring $A = R/I$ is **Gorenstein** *of socle degree s if and only if $I^\perp = \langle F \rangle$ for some polynomial F of degree s.*

Fact: $G = R/J$, with $J^{\perp} = \langle F \rangle$, is a **Gorenstein cover** of $A = R/I$ if and only if $I^{\perp} \subset J^{\perp}$.

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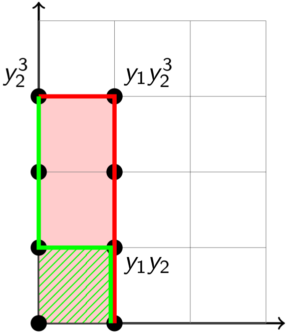

$$I^{\perp} = \langle y_1 y_2, y_2^3 \rangle \subset \langle y_1 y_2^3 \rangle = J^{\perp}$$
$$\mathrm{gcl}(R/I) \leq \ell(R/J) - \ell(R/I) = 2$$

Question: How do we know when a Gorenstein cover is **minimal**?

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Definition

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$$K_F = (I^\perp :_R J^\perp).$$

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Proposition

Let $A = R/I$ be a local Artin algebra and a Gorenstein cover $G = R/J$ of A, with $J = \operatorname{Ann} F$. Then,

(i) $I^\perp = K_F \circ F$,

(ii) $\ell(G) - \ell(A) = \ell(R/K_F)$.

Therefore, $\mathrm{gcl}(A) \leq \ell(R/K_F)$ for any Gorenstein cover $G = R/\operatorname{Ann} F$ and $\mathrm{gcl}(A) = \ell(R/K_F)$ whenever $G = R/\operatorname{Ann} F$ is a minimal cover.

I	$I^\perp$	F	K_F	$\mathrm{HF}_{R/I}$	$\mathrm{HF}_{R/J}$	gcl(A)
x_1^2, x_1x_2, x_2^4	y_1, y_2^3	$y_1^2+y_2^4$	x_1, x_2	1,2,1,1	1,2,1,1,1	1
$x_1^2, x_1x_2^2, x_2^4$	y_1y_2, y_2^3	$y_1y_2^3$	x_1, x_2^2	1,2,2,1	1,2,2,2,1	2

Definition

$A = R/I$ is Teter if $I \subseteq \mathbf{m}^2$ and there exists an Artin Gorenstein ring $G = R/J$ such that $A \cong G/Soc(G)$ and we call G the Teter cover of A.

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Theorem (Elias-Silva 17')

Let $A = R/I$, $I \subseteq \mathbf{m}^2$, be an Artin ring with maximal ideal $\mathbf{n}$, socle degree $s - 1 \geq 1$ and $\operatorname{embd}(A) > 1$. Then the following conditions are equivalent:

1. *A is a Teter ring,*
2. $\operatorname{gcl}(A) = 1$,
3. *there exists a degree s polynomial $F \in S$ such that $I^\perp = (x_1, \dots, x_n) \circ F$,*
4. *there exists an epimorphism of A-modules $I^\perp \longrightarrow \mathbf{n}$.*

In particular, if A is a Teter ring then the Cohen-Macaulay type of A is n.

Example (Minimal Gorenstein covers with non-unique Hilbert function)

$A = \mathbf{k}[\![x_1, x_2]\!]/\left(x_1^2, x_1x_2^2, x_2^4\right)$, $\mathrm{HF}_A = \{1, 2, 2, 1\}$, $I^\perp = \langle y_1y_2, y_2^3\rangle$, $\tau(A) = 2$, $\mathrm{embd}(A) = 2$.

A is clearly not Gorenstein and, by Elias-Silva characterization, we can also deduce that it is not Teter. Therefore, $\mathrm{gcl}(A) = 2$.
G_1, G_2 are minimal Gorenstein covers of socle degree 4 and 5, respectively:

- $G_1 = R/J_1$, $J_1^\perp = \langle y_1y_2^3\rangle$, $\mathrm{HF}_{G_1} = \{1, 2, 2, 2, 1\}$;
- $G_2 = R/J_2$, $J_2^\perp = \langle y_1^2y_2 + y_2^5\rangle$, $\mathrm{HF}_{G_2} = \{1, 2, 2, 1, 1, 1\}$

$\ell(G_1) - \ell(A) = \ell(G_2) - \ell(A) = 2$.

$K_{F_1} = K_{F_2} = (x_1, x_2^2)$.

Theorem (Elias-H. 17')

Let $A = R/I$ be an Artin ring with maximal ideal $\mathbf{n}$ *and socle degree $s-1 \geq 1$. We assume that A is neither Gorenstein nor Teter, $I \subseteq \mathbf{m}^5$ and* $\mathrm{char}(\mathbf{k}) \neq 2$. *Then the following conditions are equivalent:*

(i) *$gcl(A) = 2$,*

(ii) *after a linear isomorphism of R there exists a polynomial $F \in S$ of degree s or $s+1$ such that $I^\perp = (x_1, \dots, x_{n-1}, x_n^2) \circ F$,*

(iii) *there exists an epimorphism of A-modules $f : I^\perp \longrightarrow \mathfrak{q}$, where $\mathfrak{q}$ is a self-dual ideal of A by means of an isomorphism satisfying Teter's condition and $\ell(A/\mathfrak{q}) = 2$.*

In particular, if $gcl(A) = 2$ then the Cohen-Macaulay type of A is n.

Proposition (Elias-H.'17)

Let $A = R/I$ be an Artin ring such that $\mathrm{gcl}(A) \leq 2$. If $G = R/J$ is a minimal Gorenstein cover of A, then

(i) $\mathrm{embd}(G) = \mathrm{embd}(A)$,

(ii) *if $A = R/I$ with $\dim(R) = \mathrm{embd}(G) = \mathrm{embd}(A)$ and F is a generator of $J^\perp$, $G = R/J$, then $I \subset K_F$ and*

$$I^2 \subset J \subset I.$$

Moreover, after a linear isomorphism of R we may assume:

$$K_F = \begin{cases} R & \text{if} \quad \mathrm{gcl}(A) = 0 \\ \mathbf{m} & \text{if} \quad \mathrm{gcl}(A) = 1 \\ (x_1, \dots, x_{n-1}, x_n^2) & \text{if} \quad \mathrm{gcl}(A) = 2 \end{cases}$$

Definition (Integral of a module with respect to an ideal)

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Example

$I^\perp = \langle y_1 y_2, y_2^3 \rangle$,

$$\int_{\mathbf{m}^2} I^\perp = \langle y_1^3, y_1^2 y_2, y_1 y_2^3, y_2^5 \rangle.$$

Proposition

Given a ring $A = R/I$ of Gorenstein colength t and a minimal Gorenstein cover $G = R/\operatorname{Ann} F$ of A,

(i) $F \in \int_{\mathbf{m}^t} I^\perp$;

(ii) *for any $H \in \int_{\mathbf{m}^t} I^\perp$, the condition $I^\perp \subset \langle H \rangle$ does not depend on the representative of the class $\overline{H}$ in $\dfrac{\int_{\mathbf{m}^t} I^\perp}{I^\perp}$.*

In particular, any $F' \in \int_{\mathbf{m}^t} I^\perp$ such that $\overline{F'} = \overline{F}$ in $\dfrac{\int_{\mathbf{m}^t} I^\perp}{I^\perp}$ defines the same minimal Gorenstein cover $G = R/\operatorname{Ann} F$.

Theorem

Let $A = R/I$ be an Artin ring of Gorenstein colength t. There exists a quasi-projective sub-variety $MGC^n(A)$ of $\mathbb{P}_{\mathbf{k}}\left(\frac{\int_{\mathbf{m}^t} I^\perp}{I^\perp}\right)$, whose set of closed points are the points $[\overline{F}]$, $\overline{F} \in \int_{\mathbf{m}^t} I^\perp / I^\perp$, such that $G = R/\operatorname{Ann} F$ is a minimal Gorenstein cover of A.

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Definition

Given an Artin ring $A = R/I$ of Gorenstein colength t, we call $MGC^n(A)$ the minimal Gorenstein covers variety associated to A.

A closed point $[F]$ of $MGC^n(A)$ corresponds to a minimal Gorenstein cover $G = R/\operatorname{Ann} F$ of A.

Consider A such that $\mathrm{gcl}(A) = t$ and an ideal K of R such that $\ell(R/K) = t$.

Sketch of the algorithm to compute $MGC^n(A)$

For any n between $\mathrm{embd}(A)$ and $\ell(A) - \tau(A) + gcl(A) - 1$:

1. $\mathbf{k}$-basis $\overline{F}_1, \dots, \overline{F}_h$ of $\int_{\mathbf{m}^t} I^\perp / I^\perp$.
2. $F = a_1 F_1 + \cdots + a_h F_h$.
3. ideal $\mathfrak{b} \subseteq \mathbf{k}[a_1, \dots, a_h]$ such that $K \circ F \subseteq I^\perp$ for all K.
4. ideal $\mathfrak{a} \subseteq \mathbf{k}[a_1, \dots, a_h]$ such that $K \circ F \subsetneq I^\perp$ for all K.
5. $MGC^n(A) = V_+(\mathfrak{b}) \setminus \bigcap_K V_+(\mathfrak{a})$.

ALGORITHM TO COMPUTE $MGC(A)$ WHEN $\mathrm{gcl}(A) = 1$

INPUT:

- $\mathbf{k}$-basis $b_1, \ldots, b_t$ of the inverse system $I^\perp$;
- polynomials $F_1, \ldots, F_h$ such that $\overline{F_1}, \ldots, \overline{F_h}$ is a $\mathbf{k}$-basis of $\int_{\mathbf{m}} I^\perp / I^\perp$.

OUTPUT:

- ideal $\mathfrak{a} \subset \mathbf{k}[a_1, \cdots, a_h]$ such that $MCG(A) = \mathbb{P}_{\mathbf{k}}^{h-1} \backslash V_+(\mathfrak{a})$.

STEPS:

1. Define $F = a_1 F_1 + \cdots + a_h F_h$, where $a_1, \ldots, a_h$ are variables in $\mathbf{k}$.
2. Build matrix $A = \left(\mu_j^\alpha\right)_{1 \leq |\alpha| \leq s+1, 1 \leq j \leq t}$, where $x^\alpha \circ F = \sum_{j=1}^t \mu_j^\alpha b_j$.
3. Compute the ideal $\mathfrak{a}$ generated by all t-minors of A.

Example ($MGC(A)$ in Gorenstein colength 1)

$A = \mathbf{k}[\![x_1, x_2]\!]/(x_1^2, x_1x_2, x_2^4)$, $\operatorname{gcl}(A) = 1$.

$\mathbb{P}_{\mathbf{k}}(\int_{\mathbf{m}} I^{\perp}/I^{\perp}) = \mathbb{P}^2_{\mathbf{k}}$, a closed point $p = (a_1 : a_2 : a_3) \in \mathbb{P}^2_{\mathbf{k}}$ corresponds to a polynomial $F = a_1 y_2^4 + a_2 y_1 y_2 + a_3 y_1^2 \in \int_{\mathbf{m}} I^{\perp}/I^{\perp}$.

Output of the algorithm: $\mathfrak{a} = (a_1 a_3)$, hence

$$MCG(A) = \mathbb{P}^2_{\mathbf{k}} \backslash V_+(a_1 a_3).$$

Example ($MGC(A)$ in Gorenstein colength 2)

$A = \mathbf{k}[\![x_1, x_2]\!]/(x_1^2, x_1x_2^2, x_2^4)$, gcl$(A) = 2$.

$\mathbb{P}_{\mathbf{k}}(\int_{\mathbf{m}^2} I^{\perp}/I^{\perp}) = \mathbb{P}^6_{\mathbf{k}}$, a closed point $p = (a_1 : a_2 : a_3 : b_1 : b_2 : b_3 : b_4)$ corresponds to a polynomial
$F = a_1y_2^4 + a_2y_1y_2^2 + a_3y_1^2 + b_1y_1^2y_2 + b_2y_1y_2^3 + b_3y_2^5 + b_4y_1^3$.

Output of the algorithm:

- $\mathfrak{b} = (b_4) \subset \mathbf{k}[a_1, a_2, a_3, b_1, b_2, b_3, b_4]$;
- $\mathfrak{a} = \left(b_2^2 - b_1b_3\right) \subset \mathbf{k}[a_1, a_2, a_3, b_1, b_2, b_3, b_4]$.

Since $MGC(A) \subset V_+(\mathfrak{b}) = \mathbb{P}^5_{\mathbf{k}}$, a closed point
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$$MCG(A) = \mathbb{P}^5_{\mathbf{k}} \backslash V_+(b_2^2 - b_1b_3)$$

and $K_F = (x_1, x_2^2)$.

What is going on for gcl(A) ≥ 3?

Example (Admissible K_F in case gcl(A) $= 3$ and $n = 2$.)

$$G_t = \mathbf{k}[\![x_1, x_2]\!]/(x_1^t, x_2^t), \quad t \geq 5,$$

with symmetric Hilbert function $\mathrm{HF}_{G'} = \{1, 2, \ldots, t, t-1, \ldots, 1\}$ and socdeg$(G_t) = 2t - 2$, is a minimal Gorenstein cover of two non isomorphic rings of colength 3:

1. $A_{1,t} = (x_1, x_2)^2 \circ G_t$;
2. $A_{2,t} = (x_1, x_2^3) \circ G_t$.

There are two non-isomorphic admissible K_F:

1. $K_1 = (x_1, x_2)^2$, $\mathrm{HF}_{R/K_1} = \{1, 2\}$;
2. $K_2 = (x_1, x_2^3)$, $\mathrm{HF}_{R/K_2} = \{1, 1, 1\}$.

Questions we would like to answer:

- There always exist a minimal Gorenstein cover $G = R/J$ of $A = R/I$ such that $I^2 \subset J \subset I$?

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Questions we would like to answer:

- There always exist a minimal Gorenstein cover $G = R/J$ of $A = R/I$ such that $I^2 \subset J \subset I$?
- $\mathrm{embd}(G) = \mathrm{embd}(A)$?
- Explicit computation of the Gorenstein colength for, at least, certain families of Artin local rings.
- Explicit computation of $MGC(A)$ for higher colengths.

Moltes gràcies!

EXTRAS

Definition

Consider two **k**-algebras $A_i = \mathbf{k}[\![x_1, \ldots, x_n]\!]/I_i$, for $i = 1, 2$. We say that $\varphi : A_1 \longrightarrow A_2$ is an **analytic k-algebra morphism** if

1. $\varphi|_{\mathbf{k}} = Id$, and
2. φ is a ring morphism.

Definition

Consider a **k**-algebra morphism $\varphi : A_1 \longrightarrow A_2$. We say that φ is an **analytic k-algebra isomorphism** if exists a morphism $\psi : A_2 \longrightarrow A_1$ such that $\varphi \circ \psi = Id_{A_2}$ and $\psi \circ \varphi = Id_{A_1}$. This will be denoted by $A_1 \cong_{\varphi} A_2$.

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- Can a Gorenstein ring be Gorenstein cover of non-isomorphic rings of Gorenstein colength 2? ✓

Example (Non-unique minimal Gorenstein covers)

$A = \mathbf{k}[\![x_1, x_2, x_3]\!]/\left(x_1x_2, x_1x_3, x_2x_3, x_2^2, x_3^2 - x_1^3\right)$, $\mathrm{HF}_A = \{1, 3, 1, 1\}$, $I^\perp = \langle x_3^2 + x_1^3, x_2\rangle$, $\tau(A) = 2$, $\mathrm{embd}(A) = 3$. A is not Gorenstein nor Teter.

i	0	1	2	3	4	5
$\mathrm{HF}_A(i)$	1	3	1	1	0	0
$\mathrm{HF}_G(i)$	1	3	1	1	1	1
	1	3	2	1	1	0

$$J_1 = \left(x_1x_2, x_2x_3, x_2^2 - x_1^4, x_1^2x_3, x_3^2 - x_1x_3 - x_1^3\right), J_1^\perp = \langle y_2^2 + y_1y_3^2 + y_3^3 + y_1^4\rangle$$

$$J_2 = \left(x_1x_2, x_2x_3, x_2^2 - x_1^4, x_1^2x_3, x_3^2 - x_1^3\right), J_2^\perp = \langle y_1^4 + y_1y_3^2 + y_2^2\rangle$$

$\mathrm{HF}_{R/J_1} = \mathrm{HF}_{R/J_2} = \{1, 3, 2, 1, 1\}$, $K_{F_1} = K_{F_2} = (x_1, x_2, x_3^2)$.

Example (Non-isomorphic base rings of a minimal Gorenstein cover)

$$G = \mathbf{k}[\![x_1, x_2, x_3]\!]/\left(x_3^2, x_1x_2, x_1x_3, x_2^3, x_1^3 + 3x_2^2x_3\right),$$

with Hilbert function $\{1,3,3,1\}$. This ring has inverse system $J^\perp = \langle y_2^2y_3 - y_1^3\rangle$ and contains the following R-modules:

(i) $(x_2 - x_1, x_3, x_2^2) \circ J^\perp = \langle y_1^2 + y_2y_3, y_2^2\rangle = I_1^\perp$;

(ii) $(x_1 + x_2, x_2 + x_3, x_3^2) \circ J^\perp = \langle y_1^2 - y_2y_3, y_2y_3 + y_2^2\rangle = I_2^\perp$;

(iii) $(x_1, x_2, x_3^2) \circ J^\perp = \langle y_1^2, y_2y_3\rangle = I_3^\perp$;

(iv) $m \circ J^\perp = \langle y_1^2, y_2y_3, y_2^2\rangle = I^\perp$.

$A_1 = R/I_1$, $A_2 = R/I_2$ and $A_3 = R/I_3$ are non-isomorphic rings with Hilbert function $\{1,3,2\}$ and Gorenstein colength 2.

$A = R/I$ has Hilbert function $\{1,3,3\}$ and Gorenstein colength 1.

Example

$A = \mathbf{k}[\![x_1, x_2]\!]/\left(x_1^2, x_1x_2^2, x_2^4\right)$, $I^\perp = \langle y_1y_2, y_2^3\rangle$. $F_1 = y_1y_2^3$ and $F_2 = y_1^2y_2 + y_2^5$ generate inverse systems of two non-isomorphic minimal covers of A. We have epimorphisms:

$$\begin{array}{rrcl} \delta_{F_1}: & I^\perp & \longrightarrow & \mathfrak{q} = (x_1, x_2^2)/I \\ & y_1y_2 & \longmapsto & \overline{x_2^2} \\ & y_2^3 & \longmapsto & \overline{x_1} \end{array}$$

$$\begin{array}{rrcl} \delta_{F_2}: & I^\perp & \longrightarrow & \mathfrak{q} = (x_1, x_2^2)/I \\ & y_1y_2 & \longmapsto & \overline{x_1} \\ & y_2^3 & \longmapsto & \overline{x_2^2} \end{array}$$

$\mathfrak{q} = (x_1, x_2^2)/I$ is a self-dual ideal of A. Also $\ell(A/\mathfrak{q}) = \ell(K_{F_1}^\perp) = 2$.

Proposition

Let $A = R/I$ be a non-Gorenstein local Artin ring of socle degree s. Then $\mathrm{gcl}(A) = 1$ if and only if there exist a polynomial $F = \sum_{j=1}^{h} a_j F_j \in \int_{\mathbf{m}} I^{\perp}$, where $\overline{F_1}, \ldots, \overline{F_h}$ is a $\mathbf{k}$-basis of $\frac{\int_{\mathbf{m}} I^{\perp}}{I^{\perp}}$, such that $\dim_{\mathbf{k}}(\mathbf{m} \circ F) = \dim_{\mathbf{k}} I^{\perp}$.

Proposition

Given a non-Gorenstein non-Teter local Artin ring $A = R/I$, $\mathrm{gcl}(A) = 2$ if and only if there exist a polynomial $F = \sum_{i=1}^{2} \sum_{j=1}^{h} a_j^i F_j^i \in \int_{\mathbf{m}^2} I^{\perp}$, where $\overline{F_1^i}, \ldots, \overline{F_h^i}$ is a $\mathbf{k}$-basis of $\frac{\int_{\mathbf{m}^i} I^{\perp}}{\int_{\mathbf{m}^{i-1}} I^{\perp}}$, $i = 1, 2$, such that $(L_1, \ldots, L_{n-1}, L_n^2) \circ F = I^{\perp}$ for suitable independent linear forms L_i.

ALGORITHM TO COMPUTE $MGC(A)$ WHEN $\mathrm{gcl}(A) = 2$

$A = R/I$ Artin local ring of socle degree s and $n \geq 2$.

INPUT:

- **k**-basis $b_1, \dots, b_t$ of the inverse system $I^\perp$ obtained by the integration method;
- $F_1^1, \dots, F_{h_1}^1$ such that $\overline{F_1^1}, \dots, \overline{F_{h_1}^1}$ is a **k**-basis of $\int_{\mathbf{m}} I^\perp / I^\perp$.
- $F_1^2, \dots, F_{h_2}^2$ such that $\overline{F_1^2}, \dots, \overline{F_{h_2}^2}$ is a **k**-basis of $\int_{\mathbf{m}^2} I^\perp / \int_{\mathbf{m}} I^\perp$.

OUTPUT:

- -1, if all saturation ideals are R;
- The index of the first minor that provides a non-empty variety, otherwise.

STEPS:

1. Define $F = \sum_{i=1}^{h_1} a_i^1 F_i^1 + \sum_{i=1}^{h_2} a_i^2 F_i^2$, where $a_1^1, \dots, a_{h_1}^1, a_1^2, \dots, a_{h_2}^2$ are variables in **k** and $v = (v_1, \dots, v_n)$.
2. Build matrix $A = (\mu_j^i)_{1 \leq i \leq n, 1 \leq j \leq t+h_1}$, where $x_i \circ F = \sum_{j=1}^{t} \mu_j^i b_j + \sum_{j=t+1}^{t+h_1} \mu_j^i F_j^1$.
3. Build matrix $B = \left(\begin{array}{c|c} A_2 & v \end{array} \right)$ as an horizontal concatenation of $A_2 = (\mu_j^i)_{1 \leq i \leq n, t+1 \leq j \leq t+h_1}$ and the column vector v.
4. Compute the ideal I_2 generated by all minors of order 2 of B.
5. Build matrix $V = (\rho_j^{k,l})$, where $(v_l x_k - v_k x_l) \circ F = \sum_{j=1}^{t} \rho_j^{k,l} b_j$ and $\rho_j^{k,l} = v_l \mu_j^k - v_k \mu_j^l$ for any $1 \leq k < l \leq n$.
6. Build matrix U as a vertical concatenation of V and $x^\alpha \circ F = \sum_{j=1}^{t} \mu_j^\alpha b_j$, where $2 \leq |\alpha| \leq s+1$.
7. Compute the ideal I_t generated by all minors $G_1, \dots, G_r$ of order t of U.
8. Compute the saturation ideal $(I_2 : G^\infty)$.