Topological Quantum Field Theories and their application to Hodge theory

Ángel González Prieto

Universidad Complutense de Madrid

Joint work with
Marina Logares and Vicente Muñoz

arXiv:1709.05724

6th April 2018

Monoidality

A **monoidal category** is a category $\mathcal{C}$ with a distinguished object $I \in \mathcal{C}$ and a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}.$$

- $(a \otimes b) \otimes c \overset{\cong}{\to} a \otimes (b \otimes c)$.
- $I \otimes a \overset{\cong}{\to} a$ and $a \otimes I \overset{\cong}{\to} a$.
- (Symmetric) $B_{a,b} : a \otimes b \to b \otimes a$ with $B_{b,a} \circ B_{a,b} = id_{a \otimes b}$.

Examples

- R-**Mod** with tensor product and R.
- **Set** with cartesian product and $\star$.

The category of bordisms

Let $n \geq 1$. The category $\mathbf{Bord}_n$ is:

- **Objects**: $(n-1)$-dimensional closed oriented manifolds (maybe empty).
- **Morphisms**: $W : X_1 \to X_2$ is a n-dimensional compact manifold W such that $\partial W = X_1 \sqcup \overline{X}_2$, up to boundary preserving diffeomorphism (**oriented bordism**).
- **Composition**: Gluing of bordisms.

It is a symmetric monoidal category with disjoint union.

Topological Quantum Field Theories

Monoidal functor

A functor $F : (\mathcal{C}, \otimes_{\mathcal{C}}) \to (\mathcal{D}, \otimes_{\mathcal{D}})$ is said monoidal if:

- $I_{\mathcal{D}} \stackrel{\cong}{\to} F(I_{\mathcal{C}})$.
- $\Delta_{a,b} : F(a) \otimes_{\mathcal{D}} F(b) \stackrel{\cong}{\to} F(a \otimes_{\mathcal{C}} b)$.
- (Symmetric) $\Delta_{F(b),F(a)} \circ B_{F(a),F(b)} = F(B_{a,b}) \circ \Delta_{a,b}$.

TQFT

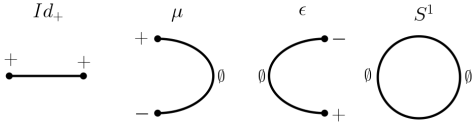
A **TQFT** is a monoidal symmetric functor

$$Z : \mathbf{Bord}_n \to k\text{-}\mathbf{Vect}.$$

Classification of TQFTs for $n = 1$

$$1\text{-}TQFT \Longleftrightarrow k\text{-}\mathbf{Vect}_0$$

- **Objects:** $\{\emptyset, +, -\}$.
- **Morphisms**



Canonical form

$$Z(+) = V \qquad Z(-) = V^*$$

$$Z(\mu) = \mathrm{ev} : V \otimes V^* \to k \qquad Z(\epsilon) = \mathrm{coev} : k \to V^* \otimes V$$

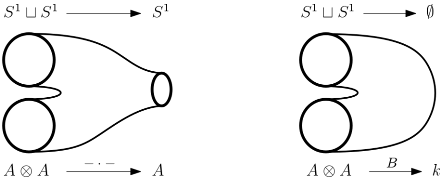
$$Z(S^1)(1) = \dim(V) < \infty$$

Classification of TQFTs for $n = 2$

$$2\text{-}TQFT \Longleftrightarrow \text{Frobenius algebras}$$

Definition: A Frobenius algebra A is a commutative finite type k-algebra with an non-degenerate bilinear form B such that

$$B(ab, c) = B(a, bc).$$



Higher dimensions: Lurie's cobordism hypothesis (2009).

Relaxing monoidality

Problem: Duality implies that TQFT must be finite dimensional.

Lax monoidal TQFT

A **lax monoidal TQFT** is a lax monoidal symmetric functor

$$Z : \mathbf{Bord}_n \to R\text{-}\mathbf{Mod}.$$

Lax monoidality

The map

$$\Delta_{X_1,X_2} : Z(X_1) \otimes Z(X_2) \to Z(X_1 \sqcup X_2)$$

is no longer an isomorphism.

Physical inspiration of TQFT

Let $\mathcal{C}$ be a category with final object $\star$ and pullbacks (category of fields).

$$Z : \mathbf{Bord}_n \xrightarrow{\text{Field theory}} \mathrm{Span}(\mathcal{C}) \xrightarrow{\text{Quantization}} R\text{-}\mathbf{Mod}$$

The category of spans

- **Objects:** $\mathrm{Obj}(\mathrm{Span}(\mathcal{C})) = \mathrm{Obj}(\mathcal{C})$.
- **Morphisms:** A morphism $S : c_1 \to c_2$ is a span in $\mathcal{C}$

$$\begin{array}{ccc} & d & \\ \swarrow & & \searrow \\ c_1 & & c_2 \end{array}$$

Physical inspiration of TQFT

$$Z : \mathbf{Bord}_n \xrightarrow{\text{Field theory}} \mathrm{Span}(\mathcal{C}) \xrightarrow{\text{Quantization}} R\text{-}\mathbf{Mod}$$

Field Theory

Let $\mathcal{G} : \mathbf{Diff}_c^{or} \to \mathcal{C}$ be a monoidal contravariant functor sending pushforwards into pullbacks and define

$$\mathcal{F}_{\mathcal{G}} : \mathbf{Bord}_n \longrightarrow \mathrm{Span}(\mathcal{C}).$$

- **Objects:** $\mathcal{F}_{\mathcal{G}}(X) = \mathcal{G}(X)$.
- **Morphisms:** Given $W : X_1 \to X_2$, its image is the span

$$\mathcal{G}(X_1) \xleftarrow{\mathcal{G}(i_1)} \mathcal{G}(W) \xrightarrow{\mathcal{G}(i_2)} \mathcal{G}(X_2).$$

$\mathcal{C}$-Algebra

A $\mathcal{C}$**-algebra** $\mathcal{A}$ is a pair of functors:

$$A : \mathcal{C}^{op} \to \textbf{Ring} \qquad B : \mathcal{C} \to A(\star)\text{-}\textbf{Mod}$$

- They agree on objects: $A(c) = B(c)$ for all $c \in \mathcal{C}$.
- Beck-Chevaley condition: For a pullback diagram

$$\begin{array}{ccc} d & \xrightarrow{g'} & c_1 \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ c_2 & \xrightarrow[g]{} & c \end{array} \qquad A(g) \circ B(f) = B(f') \circ A(g')$$

$$\mathcal{A}_c = A(c) \in \textbf{Ring} \qquad f^* = A(f) \qquad f_! = B(f)$$

Quantization

Given a $\mathcal{C}$-algebra $\mathcal{A}$, we define

$$\mathcal{Q}_{\mathcal{A}} : \mathrm{Span}(\mathcal{C}) \to \mathcal{A}_\star\text{-}\mathbf{Mod}.$$

- **Objects:** $\mathcal{Q}_{\mathcal{A}}(c) = \mathcal{A}_c$ for $c \in \mathcal{C}$.
- **Morphisms:** Given a span $S : c_1 \xleftarrow{f} d \xrightarrow{g} c_2$, we define

$$\mathcal{Q}_{\mathcal{A}}(S) = g_! \circ f^* : \mathcal{A}_{c_1} \xrightarrow{f^*} \mathcal{A}_d \xrightarrow{g_!} \mathcal{A}_{c_2}$$

Construction of TQFT

$$Z : \mathbf{Bord}_n \xrightarrow{\mathcal{F}_{\mathcal{G}}} \mathrm{Span}(\mathcal{C}) \xrightarrow{\mathcal{Q}_{\mathcal{A}}} R\text{-}\mathbf{Mod}$$

Representation varieties

Let G be a complex algebraic group and Γ a finitely generated group. The **representation variety** is

$$\mathfrak{X}_G(\Gamma) = \operatorname{Hom}(\Gamma, G).$$

If $\Gamma = \pi_1(M)$ we denote $\mathfrak{X}_G(M) := \mathfrak{X}_G(\pi_1(M))$.

Algebraic structure: $\Gamma = \langle \gamma_1, \dots, \gamma_s \mid R_\alpha(\gamma_1, \dots, \gamma_s) = 1 \rangle$.
We have an identification

$$\begin{array}{rccc} \psi : & \operatorname{Hom}(\Gamma, G) & \longrightarrow & G^s \\ & \rho & \mapsto & (\rho(\gamma_1), \dots, \rho(\gamma_s)) \end{array}$$

with the algebraic variety

$$\operatorname{Im} \psi = \left\{ (g_1, \dots, g_s) \in G^s \mid R_\alpha(g_1, \dots, g_s) = 1 \right\}.$$

Mixed Hodge structure

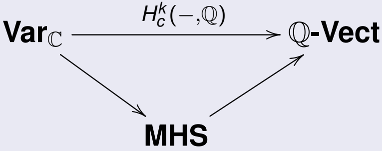
Let X be a complex algebraic variety. There is a natural double finite filtration

$$0 \subseteq \ldots \subseteq W_s H_c^k(X;\mathbb{Q}) \subseteq W_{s+1} H_c^k(X;\mathbb{Q}) \subseteq \ldots \subseteq H_c^k(X;\mathbb{Q})$$

$$0 \supseteq \ldots \supseteq F^p H_c^k(X;\mathbb{Q})_{\mathbb{C}} \supseteq F^{p+1} H_c^k(X;\mathbb{Q})_{\mathbb{C}} \supseteq \ldots \supseteq H_c^k(X;\mathbb{Q})_{\mathbb{C}}$$

called the **mixed Hodge structure.**

Deligne's theorem



Deligne-Hodge polynomial

$$H_c^{k;p,q}(X) = Gr_F^p \, Gr_{p+q}^W \, H_c^k(X;\mathbb{Q}) \qquad h_c^{k;p,q}(X) = \dim H_c^{k;p,q}(X)$$

Deligne-Hodge polynomial

$$e(X) = \sum_k \sum_{p,q} (-1)^k h_c^{k;p,q}(X)\, u^p v^q \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}].$$

Generalized Euler characteristic

$$e(X_1 \sqcup X_2) = e(X_1) + e(X_2) \qquad e(X_1 \times X_2) = e(X_1)\, e(X_2).$$

Defines a ring homomorphism $e : \mathrm{K}\mathbf{Var}_{\mathbb{C}} \to \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$.

Example: Smooth projective varieties

Hodge's theorem

If X is a smooth projective variety

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

$$0 = W_{k-1} \subseteq W_k = H^k(X;\mathbb{Q}) \qquad F^p = \bigoplus_{s\geq p} H^{s,k-s}(X)$$

$$H_c^{k;p,q}(X) = H^{p,q}(X) \qquad e(X) = \sum_{p,q} (-1)^{p+q} h^{p,q}(X)\, u^p v^q$$

- $e(\mathbb{C}P^n) = 1 + uv + \ldots + u^n v^n \Rightarrow e(\mathbb{C}^n) = u^n v^n$.
- $e(\mathrm{Proj\ curve}) = 1 - gu - gv + uv$.

Arithmetic method

Problem

Compute $e(\mathfrak{X}_G(\Sigma_g))$ for Σ_g the genus g surface.

Based on Katz' theorem of polynomial counting.

- Hausel and Rodriguez-Villegas (2008). $G = GL(n, \mathbb{C})$, arbitrary g. Twisted.
- Hausel, Letellier and Rodrigez-Villegas (2011). $G = GL(n, \mathbb{C})$, arbitrary g. Generic semi-simple marked points.
- Mereb (2015). $G = SL(n, \mathbb{C})$, arbitrary g. Twisted.

In terms of generating functions.

Geometric method

Based on Hodge monodromy representation.

- Logares, Muñoz and Newstead (2013). $G = SL(2, \mathbb{C})$, $g = 1, 2$. At most 1 marked point.
- Logares and Muñoz (2014). $G = SL(2, \mathbb{C})$, $g = 1$. At most 2 marked points.
- Martínez and Muñoz (2016). $G = SL(2, \mathbb{C})$, arbitrary g.
- Martínez (2017). $G = PGL(2, \mathbb{C})$, arbitrary g.
- Baraglia and Hekmati (2017). $G = GL(2, \mathbb{C}), GL(3, \mathbb{C})$, $SL(2, \mathbb{C}), SL(3, \mathbb{C})$, arbitrary g.

Explicit expressions.

Field theory for representation varieties

Goal

Construct a TQFT that computes $e(\mathfrak{X}_G(\Sigma_g))$.

Remark: Instead of $\mathbf{Bord}_n$, we will use $\mathbf{Bordp}_n$, the category of n-dimensional (pairs) bordisms with a marked finite subset.

$$Z : \mathbf{Bordp}_n \xrightarrow[\mathcal{F}_{\mathcal{G}}]{\text{Field theory}} \mathrm{Span}(\mathbf{Var}_{\mathbb{C}}) \xrightarrow[\mathcal{Q}_{\mathcal{A}}]{\text{Quantization}} R\text{-}\mathbf{Mod}$$

We define $\mathcal{G} : \mathbf{Diffp}_c^{or} \to \mathbf{Var}_{\mathbb{C}}$ by

$$\mathcal{G}(M, A) = \mathrm{Hom}\,(\underbrace{\Pi(M, A)}_{\text{Fund. groupoid}}, G),$$

for M a compact manifold and $A \subseteq M$ finite.

Quantization via mixed Hodge modules

Saito's mixed Hodge modules

Complex algebraic variety X $\Longrightarrow$ Abelian monoidal category $\mathcal{M}_X$ $\Longrightarrow$ Ring $\mathrm{K}\mathcal{M}_X$

- $\mathcal{M}_X$ contains variations of Hodge structures on X.
- $\mathcal{M}_\star = \mathbf{MHS}$, category of mixed Hodge structures.
- Hodge monodromy representation can be understood in this context.

Quantization for representation varieties

Properties

- Every $\mathrm{K}\mathcal{M}_X$ has a natural $\mathrm{K}\mathbf{MHS}$-module structure.
- For $f : X \to Y$ regular we have $\mathrm{K}\mathbf{MHS}$-module morphisms

$$f_*, f_! : \mathrm{K}\mathcal{M}_X \to \mathrm{K}\mathcal{M}_Y, \qquad f^*, f^! : \mathrm{K}\mathcal{M}_Y \to \mathrm{K}\mathcal{M}_X.$$

  f^* is also a ring homomorphism.
- Beck-Chevalley condition for pairs $(f^*, f_!)$ and $(f_*, f^!)$.
- For the projection $c_X : X \to \star$

$$(c_X)_!(1) = [H_c^\bullet(X; \mathbb{Q})].$$

For representation varieties

$\mathrm{K}\mathcal{M} = (f^*, f_!)$ is a $\mathbf{Var}_{\mathbb{C}}$-algebra with $\mathrm{K}\mathcal{M}_\star = \mathrm{K}\mathbf{MHS}$.

Main theorem

$$Z = \mathcal{Q}_{\mathrm{K}\mathcal{M}} \circ \mathcal{F}_{\mathcal{G}} : \mathbf{Bordp}_n \to \mathrm{K}\mathbf{MHS}\text{-}\mathbf{Mod}$$

Let W be a closed manifold, seen as a bordism $W : \emptyset \to \emptyset$, and $A \subseteq W$ finite. We have $Z(W, A) : \mathrm{K}\mathbf{MHS} \to \mathrm{K}\mathbf{MHS}$.

$$Z(W, A)(1) = [H_c^\bullet(\mathfrak{X}(W); \mathbb{Q})] \times [H_c^\bullet(G; \mathbb{Q})]^{|A|-1},$$

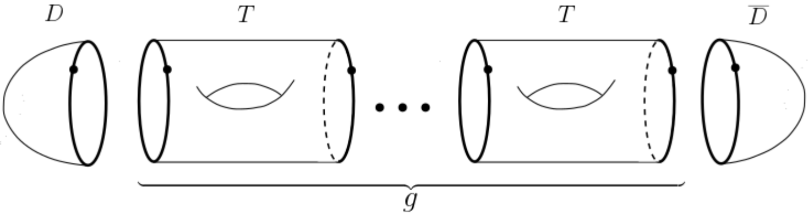
$$e(Z(W, A)(1)) = e(G)^{|A|-1} e(\mathfrak{X}(W)).$$

Theorem (G-P, Logares, Muñoz)

There exists a lax monoidal TQFT computing Deligne-Hodge polynomials of representation varieties.

TQFT in surfaces

Set $n = 2$ (surfaces). The genus g closed orientable surface Σ_g with $g + 1$ marked points can be writen as $\Sigma_g = \overline{D} \circ T^g \circ D$.



$$e(\mathfrak{X}(\Sigma_g)) = \frac{1}{e(G)^g} \left[Z(\overline{D}) \circ Z(T)^g \circ Z(D)(1) \right]$$

$$\mathcal{G}(\emptyset) = 1 \qquad \mathcal{G}(S^1) = G$$
$$\mathcal{G}(D) = \mathcal{G}(\overline{D}) = G \qquad \mathcal{G}(T) = G^4$$

Field Theory morphisms

$$\mathcal{F}_{\mathcal{G}}(D) = \left[1 \longleftarrow 1 \xrightarrow{i} G\right] \qquad \mathcal{F}_{\mathcal{G}}(\overline{D}) = \left[G \xleftarrow{i} 1 \longrightarrow 1\right]$$

$$\mathcal{F}_{\mathcal{G}}(T) = \begin{bmatrix} G & \xleftarrow{p} & G^4 & \xrightarrow{s} & G \\ g & \leftarrow\!\shortmid & (g, g_1, g_2, h) & \mapsto & hg[g_1, g_2]h^{-1} \end{bmatrix}$$

Morphisms of the TQFT

$$Z(D) = i_! \qquad Z(\overline{D}) = i^* \qquad Z(T) = s_! p^*$$

Concluding remarks

- It is enough to compute three linear maps.
- Also valid for the parabolic case and non-orientable manifolds.
- Functor of 2-categories.
- General method of construction of TQFTs.

Future work

- Explicit computations.
- Reformulation of geometric method in this framework.
- Can we remove lax monoidality?
- Is $W_G = \langle Z(T)^g \, i_!(1) \rangle_{g=0}^{\infty} \subseteq \mathrm{K}\mathcal{M}_G$ finite dimensional?
- New perspectives for mirror symmetry for character varieties.

Thank you

for your attention