Strong homology and set theory

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Abstract

Various connections between strong homology theory and set theory are discussed. The Continuum Hypothesis, the Proper Forcing Axiom, and other set-theoretic axioms imply different values of higher derived limits, and strong homology groups.

The topic we are going to talk about, has quite a long history. In [MP88] it was proved that strong homology neither is additive nor has compact supports, provided the Continuum Hypothesis (CH) is assumed. In that paper, an abelian pro-group $\mathcal{A}$ was constructed such that the groups $\varprojlim^s \mathcal{A}$, $s \geq 1$, serve as the obstructions both to additivity and having compact supports. Assuming CH, it was proved that $\varprojlim^1 \mathcal{A} \neq 0$. Later, in [Pra05], under a weaker assumption $\mathfrak{d} = \aleph_1$, it was proved that the cardinality of $\varprojlim^1 \mathcal{A}$ is quite large:

$$\left|\varprojlim{}^1 \mathcal{A}\right| = \aleph_1^{\aleph_1}.$$

In [DSV89], assuming the Proper Forcing Axiom (PFA), it was proved that

$$\varprojlim{}^1 \mathcal{A} = 0.$$

However, recently in [Ber15], it was shown that, assuming PFA,

$$\begin{aligned} \varprojlim{}^2 \mathcal{A} &\neq 0, \\ \varprojlim{}^s \mathcal{A} &= 0, s \neq 0, 2. \end{aligned}$$

An interesting result in [Tod98, Theorem 1] should be also mentioned: $\varprojlim^1 \mathcal{A} \neq 0$ implies that there is a subset X of $\mathbb{R} - \mathbb{Q}$ which is not analytic but its intersection with every compact subset of $\mathbb{R} - \mathbb{Q}$ is F_σ. Another interesting result [Far11, Theorem 1.1] uses the methods of [MP88]: the Continuum Hypothesis implies there is an outer automorphism of the Calkin algebra. Moreover, the restriction of this automorphism to any separable subalgebra is inner.

What if only ZFC (the Zermelo-Fraenkel axioms plus the Axiom of Choice) is assumed? Günther [Gün92, Example before Theorem 8] proved the following.

Let $X = \omega_1$, and let A be the subspace of limit ordinals. X is a normal Hausdorff space. It was proved that

$$\overline{H}{}^c_n(X, A; \mathbb{Z}_2) \ncong \overline{H}{}^c_n(X/A, *; \mathbb{Z}_2)$$

where $\overline{H}_n$ is strong homology, while $\overline{H}^c_n$ is strong homology with compact supports. It follows then that one (or both) of the following statements is valid:

1. $$\overline{H}^c_n(X, \mathbb{Z}_2) \ncong \overline{H}_n(X, \mathbb{Z}_2)$$

2. $$\overline{H}^c_n(A, \mathbb{Z}_2) \ncong \overline{H}_n(A, \mathbb{Z}_2)$$

Therefore, strong homology does not have compact supports.

Mardešić (see [Mar96] and [Mar00, Ch. 21.5]) constructed paracompact spaces X such that

$$\overline{H}^c_n(X) \ncong \overline{H}_n(X).$$

Lisica [Lis05] constructed a separable metric space X with $\overline{H}^c_n(X) \ncong \overline{H}_n(X)$.

The problem of additivity was finally solved in the negative in [Pra05]: it was proved that, assuming only ZFC, strong homology is **not** additive.

It could seem that strong homology is a source of mostly **negative** results. However, there are a lot of clearly **positive** results, like in [MP98, Theorem 1 and Theorem 10]:

Let X be a compact Hausdorff space, represented as a limit of finite polyhedra:

$$X = \varprojlim_{i \in \mathbf{I}} X_i.$$

Given an abelian group G and integer $m \in \mathbb{Z}$, the strong homology group $\overline{H}_m(X; G)$ has a natural (on X and G) filtration

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 = \overline{H}_m(X, G)$$

together with the following natural isomorphisms:

1. $F_1 \simeq Pext\left(\check{H}^{m+1}(X), G\right) \simeq \varprojlim^1_{i \in \mathbf{I}} H_{m+1}(X_i, G) \simeq \varprojlim^1_{i \in \mathbf{I}} Hom\left(H^{m+1}(X_i), G\right)$;
2. $F_2 \simeq Ext\left(\check{H}^{m+1}(X), G\right)$;
3. $F_2/F_1 \simeq \varprojlim_{i \in \mathbf{I}} Ext\left(H^{m+1}(X_i), G\right)$;
4. $F_3/F_1 \simeq \check{H}_m(X, G)$;
5. $F_3/F_2 \simeq Hom\left(\check{H}^m(X), G\right) \simeq \varprojlim_{i \in \mathbf{I}} \left(H^m(X_i), G\right)$,

where $\check{H}_*$ and $\check{H}^*$ are respectively Čech homology and cohomology.

Moreover, $\overline{H}_m(X, G) = 0$ if $m < 0$.

1 Pro-category

Given a category $\mathbf{C}$, consider a sequence of full embeddings

$$\mathbf{C} \overset{i}{\longrightarrow} \mathbf{Pro}\left(\mathbf{C}\right) \overset{j}{\longrightarrow} \left(\mathbf{Set}^{\mathbf{C}}\right)^{op}$$

where ji is the second Yoneda embedding

$$ji = h^{?} : \mathbf{C} \longrightarrow \left(\mathbf{Set}^{\mathbf{C}}\right)^{op},$$

and $\mathbf{Pro}\left(\mathbf{C}\right)$ is the full subcategory of $\left(\mathbf{Set}^{\mathbf{C}}\right)^{op}$ consisting of pro-representable functors:

$$\mathcal{X} \in \mathbf{Pro}\left(\mathbf{C}\right) \iff \mathcal{X} = \varinjlim_{i \in \mathbf{I}} \left(h^{X_i}\right)$$

where $\mathbf{I}$ is the small *cofiltrant* category.

Remark 1.1 ***Cofiltrant*** *means dual to* ***filtrant*** *(****filtered****,* ***filtering****):*

1. *For any $i, j \in \mathbf{I}$ there exist a $k \in \mathbf{I}$, and morphisms*

$$k \longrightarrow i, k \longrightarrow j.$$

2. *For any two morphisms*

$$i \underset{\beta}{\overset{\alpha}{\rightrightarrows}} j$$

there exists a morphism

$$\gamma : k \longrightarrow i$$

with $\alpha\gamma = \beta\gamma$.

Remark 1.2 *A (pre)poset $\mathbf{I}$ is called cofiltrant or* ***codirected*** *if the corresponding category:*

$$Hom_{\mathbf{I}}\left(x, y\right) = \begin{cases} (x, y) & if \quad x \leq y \\ \varnothing & otherwise \end{cases}$$

is cofiltrant.

Remark 1.3 *Objects in $\mathbf{Pro}\left(\mathbf{C}\right)$ can be described by diagrams*

$$\mathcal{X} : \mathbf{I} \longrightarrow \mathbf{C}$$

where $\mathbf{I}$ is a small cofiltrant category. Morphisms can be described like this:

$$Hom_{\mathbf{Pro}(\mathbf{C})}\left(\mathcal{X} : \mathbf{I} \longrightarrow \mathbf{C}, \mathcal{Y} : \mathbf{J} \longrightarrow \mathbf{C}\right) =$$

$$= \varprojlim_{j \in \mathbf{J}} \varinjlim_{i \in \mathbf{I}} Hom_{\mathbf{C}}\left(\mathcal{X}\left(i\right), \mathcal{Y}\left(j\right)\right).$$

*Using Mardešić trick [MS82, Theorem I.1.4], one can assume that $\mathbf{I}$ and $\mathbf{J}$ are cofiltrant (pre)****posets****.*

Example 1.4 *Let* $\mathbf{Fin}$ *be the category of finite sets. Then* $\mathbf{Pro}(\mathbf{Fin})$ *is equivalent to the category of* ***profinite sets****, i.e.* 0*-dimensional compact Hausdorff spaces.*

Example 1.5 *Let* $\mathbf{FinGr}$ *be the category of finite groups. Then* $\mathbf{Pro}(\mathbf{FinGr})$ *is equivalent to the category of* ***profinite groups****, i.e.* 0*-dimensional compact Hausdorff groups. The equivalence is given by the following:*

$$\begin{array}{rcl} G & \longmapsto & (G/U)_{U\in Norm(G)} \in \mathbf{Pro}(\mathbf{FinGr}), \\ \mathbf{Pro}(\mathbf{FinGr}) & \ni & (G_i)_{i\in\mathbf{I}} \longmapsto \varprojlim_i G_i, \end{array}$$

where $Norm(G)$ *is the set of open normal subgroups of* G*, and* $\varprojlim$ *is taken in the category of topological groups.*

Remark 1.6 *The ind-category* $\mathbf{Ind}(\mathbf{C})$ *is defined dually:*

$$\mathbf{C} \xrightarrow{i'} \mathbf{Ind}(\mathbf{C}) = (\mathbf{Pro}(\mathbf{C}^{op}))^{op} \xrightarrow{j'} \mathbf{Set}^{\mathbf{C}^{op}}$$

where $j'i'$ *equals the first Yoneda embedding:*

$$j'i' = h_? : \mathbf{C} \longrightarrow \mathbf{Set}^{\mathbf{C}^{op}}.$$

2 Shape theory

The main reference is [MS82].

Let $\mathcal{P}$ be the class of topological spaces having the homotopy type of a polyhedron. By the same letter we will denote the corresponding full subcategory of $\mathbf{Top}$.

Remark 2.1 $\mathcal{P}$ *can be described equivalently as the class of topological spaces having the homotopy type of an absolute neighborhood retract (*ANR*).*

Definition 2.2 *Let* X *be a space. Consider the set* $Cov(X)$ *of* ***normal*** *(numerable) coverings on* X*. The set is pre-ordered by the following:*

$$\mathcal{U} \leq \mathcal{V} \iff \mathcal{U} \text{ refines } \mathcal{V}.$$

The family of homotopy types of Čech nerves $(N\mathcal{U})_{\mathcal{U}\in Cov(X)}$ *represents an object of* $\mathbf{Pro}(H(\mathcal{P}))$ *which is called the* ***shape*** *of* X*.*

The ***pointed shape*** *is defined similarly.*

Remark 2.3 *If* X *is (Hausdorff) paracompact then the set of normal coverings can be replaced by the set of* ***all*** *coverings.*

Definition 2.4 *Given a space* X *and an abelian group* G*, let*

$$\mathbf{Pro}\text{-}H_n(X,G) := (H_n(N\mathcal{U},G))_{\mathcal{U}\in Cov(X)} \in \mathbf{Pro}(\mathbf{Ab})$$

be its ***pro-homology****.*

Definition 2.5 *Given a pointed space X, let*

$$\mathbf{Pro}\text{-}\pi_n(X) := (\pi_n(N\mathcal{U}))_{\mathcal{U}\in Cov(X)} \in \mathbf{Pro}(\mathbf{C})$$

where

$$\mathbf{C} = \begin{cases} \mathbf{Set} & \text{if} \quad n = 0 \\ \mathbf{Gr} & \text{if} \quad n = 1 \\ \mathbf{Ab} & \text{if} \quad n \geq 2 \end{cases}$$

*be its **pro-homotopy**.*

Example 2.6 *The Warsaw circle.*

Example 2.7 *Given a family $(X_a)_{\alpha\in A}$ of pointed spaces, let Y be their **bouquet**, i.e. the coproduct of X_α in the category of pointed sets. Y is naturally a subset of the product:*

$$Y \subseteq \prod_{\alpha\in A} X_\alpha.$$

*Let the **cluster** (or **wedge**, or **compact bouquet**) of the family $(X_a)_{\alpha\in A}$ be Y with the subspace topology. Denote it by*

$$Y = \overline{\bigvee_{\alpha\in A}} X_\alpha.$$

Remark 2.8 *The cluster is **not** isomorphic to the coproduct*

$$\bigvee_{\alpha\in A} X_\alpha$$

in the category $\mathbf{Top}_*$ of pointed topological spaces.

Example 2.9 *The Hawaiian ear-ring $X^{(k)}$ is the cluster of countably many copies of the k-sphere:*

$$X^{(k)} = \overline{\bigvee_{n\in\mathbb{N}}} S^k.$$

3 Strong shape theory

The main reference is [Mar00].

Definition 3.1 *Let X be a space. Consider the family of Čech nerves $(N\mathcal{U})_{\mathcal{U}\in Cov(X)}$ which represents an object of $\mathbf{Pro}(H(\mathcal{P}))$. The diagram*

$$\mathcal{U} \longmapsto N\mathcal{U} : Cov(X) \longrightarrow \mathcal{P}$$

*is commutative only up to homotopy, and does therefore **not** represent an object of $\mathbf{Pro}(\mathcal{P})$. The homotopies are homotopic up to a second order homotopy. The second order homotopies are homotopic up to a third order homotopy etc. This family of higher order homotopies defines a **coherent** homotopy commutative diagram which represents the **strong shape** of X. The morphisms between such diagrams are defined similarly, up to a coherent homotopy.*

*The **pointed strong shape** is defined similarly.*

Remark 3.2 *In fact, the objects of the strong shape category* **SSh** ***can*** *be represented as objects of* $\mathbf{Pro}(\mathcal{P})$*, i.e.* ***strictly*** *commutative diagrams, while the morphisms between such diagrams are described as coherent families of mappings. However, it is possible [Pra01] to define* **SSh** *as a full subcategory of the localization* $\mathbf{Pro}(\mathcal{P})\left[\Sigma^{-1}\right]$ *for an appropriate family* Σ *of morphisms.*

4 Strong homology

Let $\mathcal{X} \in \mathbf{Pro}(\mathcal{P})$ be given by a diagram

$$\mathcal{X} : \mathbf{I} \longrightarrow \mathcal{P}.$$

Consider a cochain bicomplex $C^{\bullet\bullet}(\mathcal{X}, G)$:

$$C^{st}(\mathcal{X}, G) = \prod_{(i_0 \to i_1 \to \ldots \to i_s) \in \mathbf{I}} C_{-t}^{sing}(X(i_s), G)$$

where $C_{\bullet}^{sing}(X(i_s), G)$ is the singular chain complex.

Definition 4.1 *Strong homology of* X *is defined as the cohomology of the total cochain complex*

$$\overline{H}_n(X, G) = H^{-n}(Tot(C^{\bullet\bullet}(\mathcal{X}), G))$$

where $\mathcal{X} \in \mathbf{Pro}(\mathcal{P})$ *represents the strong shape of* X*.*

The relative strong homology groups $\overline{H}_n(X, A; G)$ *are defined similarly.*

Remark 4.2 *Since the bicomplex* $C^{\bullet\bullet}(\mathcal{X}, G)$ *occupies the IV quadrant, it is possible that nontrivial strong homology groups* $\overline{H}_n(X, G)$ *exist for* $n < 0$*. However, it is* ***impossible*** *for compact Hausdorff spaces.*

In [Pra01], strong homology $\overline{H}_n(X, E)$ with the coefficients in a spectrum E is defined. When $E = K(G)$ is the Eilenberg - Mac Lane spectrum, there exists a natural isomorphism with strong homology from the above definition:

$$\overline{H}_n(X, E) \simeq \overline{H}_n(X, G).$$

Strong homology satisfy the seven Eilenberg-Steenrod Axioms and Axioms 8 and 9 from [Mil95]. On the category of compact metric spaces, strong homology is isomorphic to the Steenrod homology. Moreover, there exists a conditionally convergent spectral sequence [Pra89], [Pra13, Theorem 4.5]:

$$E_2^{st} = \varprojlim{}^s H_{-t}(\mathcal{X}, G) \Rightarrow \overline{H}_{-s-t}(X, G).$$

5 Additivity

In [Mil62], Milnor formulated the Additivity axiom: given a homology theory h_*, and a family $(X_\alpha)_{\alpha\in A}$ of topological spaces, there is a natural isomorphism:

$$h_*\left(\coprod_{\alpha\in A} X_\alpha\right) \simeq \bigoplus h_*\left(X_\alpha\right).$$

Remark 5.1 *The axiom is evidently valid for a* ***finite*** *family* (X_α).

Let $X^{(k)}$ be the k-dimensional Hawaiian ear-ring, i.e. the cluster of countably many copies of the k-sphere:

$$X^{(k)} = \overline{\bigvee_{n\in\mathbb{N}}} S^k,$$

and let

$$Y^{(k)} = \coprod_{n\in\mathbb{N}} X^{(k)}$$

$Y^{(k)}$ is a locally compact metric space, which can be embedded into $\mathbb{R}^{k+1}$.

Definition 5.2 *Let* $\leq$ *be the pointwise order on* $\mathbb{N}^{\mathbb{N}}$*, and let*

$$\mathbf{I} = \left(\mathbb{N}^{\mathbb{N}}, \leq\right).$$

We define the following pro-group $\mathcal{A}$*:*

$$\begin{array}{rcl} \mathcal{A} & : & \mathbf{I} \longrightarrow \mathbf{Ab}, \\ \mathcal{A}\left(f\right) & = & \displaystyle\bigoplus_{n\in\mathbb{N}} \bigoplus_{m\leq f(n)} \mathbb{Z}. \end{array}$$

Definition 5.3 *Let* $\leq^*$ *be another pre-order on* $\mathbb{N}^{\mathbb{N}}$*:*

$$f \leq^* g \iff \left|n : f\left(n\right) > g\left(n\right)\right| < \infty.$$

The dominating number $\mathfrak{d}$ *equals the cofinality of* $\mathbf{I}^* = \left(\mathbb{N}^{\mathbb{N}}, \leq^*\right)$*. The bounding number* $\mathfrak{b}$ *equals the least cardinality of an unbounded subset of* $\mathbf{I}^*$*.*

Theorem 5.4 *[MP88], [Pra05], [DSV89], [Ber15]*
Let

$$S_{m,k} := coker\left(\bigoplus_{n\in\mathbb{N}} \overline{H}_m\left(X^{(k)}\right) \to \overline{H}_m\left(Y^{(k)}\right)\right).$$

Then:

1.
$$S_{m,k} = \begin{cases} 0 & if \quad m \geq k \\ \varprojlim^{k-m} \mathcal{A} & if \quad m < k \end{cases}$$

2. *If* $\mathfrak{d} = \aleph_1$ *is assumed, then* $\varprojlim{}^1 \mathcal{A} \neq 0$. *Moreover,*

$$\left|\varprojlim{}^1 \mathcal{A}\right| = \aleph_1^{\aleph_1}.$$

3. *If* PFA *is assumed, then:*

$$\begin{aligned} \varprojlim{}^2 \mathcal{A} &\neq 0, \\ \varprojlim{}^s \mathcal{A} &= 0, s \neq 0, 2. \end{aligned}$$

Remark 5.5 *The last statement can be proved under a weaker assumption*

$$(\mathfrak{b} = \mathfrak{d} = \mathfrak{c} = \aleph_2) \,\&\diamondsuit\left(S_1^2\right).$$

References

[Ber15] Jeffrey Bergfalk. Strong homology, derived limits, and set theory. *Fundamenta Mathematicae (to appear)*, http://arxiv.org/abs/1509.09267, 2015.

[DSV89] Alan Dow, Petr Simon, and Jerry E. Vaughan. Strong homology and the proper forcing axiom. *Proc. Amer. Math. Soc.*, 106(3):821–828, 1989.

[Far11] Ilijas Farah. All automorphisms of the Calkin algebra are inner. *Ann. of Math. (2)*, 173(2):619–661, 2011.

[Gün92] Bernd Günther. The Vietoris system in strong shape and strong homology. *Fund. Math.*, 141(2):147–168, 1992.

[Lis05] Ju. T. Lisica. Strong bonding homology and cohomology. *Topology Appl.*, 153(2-3):394–447, 2005.

[Mar96] Sibe Mardešić. Nonvanishing derived limits in shape theory. *Topology*, 35(2):521–532, 1996.

[Mar00] Sibe Mardešić. *Strong shape and homology*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[Mil62] J. Milnor. On axiomatic homology theory. *Pacific J. Math.*, 12:337–341, 1962.

[Mil95] John Milnor. On the Steenrod homology theory. In *Novikov conjectures, index theorems and rigidity, Vol. 1 (Oberwolfach, 1993)*, volume 226 of *London Math. Soc. Lecture Note Ser.*, pages 79–96. Cambridge Univ. Press, Cambridge, 1995.

[MP88] S. Mardešić and A. V. Prasolov. Strong homology is not additive. *Trans. Amer. Math. Soc.*, 307(2):725–744, 1988.

[MP98] Sibe Mardešić and Andrei V. Prasolov. On strong homology of compact spaces. *Topology Appl.*, 82(1-3):327–354, 1998. Special volume in memory of Kiiti Morita.

[MS82] Sibe Mardešić and Jack Segal. *Shape theory*, volume 26 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, 1982.

[Pra89] Andrei V. Prasolov. A spectral sequence for strong homology. *Glas. Mat. Ser. III*, 24(44)(1):17–24, 1989.

[Pra01] Andrei V. Prasolov. Extraordinary strong homology. *Topology Appl.*, 113(1-3):249–291, 2001. Geometric topology: Dubrovnik 1998.

[Pra05] Andrei V. Prasolov. Non-additivity of strong homology. *Topology Appl.*, 153(2-3):493–527, 2005.

[Pra13] Andrei V. Prasolov. On the universal coefficients formula for shape homology. *Topology Appl.*, 160(14):1918–1956, 2013.

[Tod98] Stevo Todorcevic. The first derived limit and compactly F_σ sets. *J. Math. Soc. Japan*, 50(4):831–836, 1998.