Regularity of free boundaries in obstacle problems

Xavier Ros-Oton

Abstract Free boundary problems are those described by PDE that exhibit a priori unknown (free) interfaces or boundaries. Such type of problems appear in Physics, Geometry, Probability, Biology, or Finance, and the study of solutions and free boundaries uses methods from PDE, Calculus of Variations, and Geometric Measure Theory. The main mathematical challenge is to understand the regularity of free boundaries. The Stefan problem and the obstacle problem are the most classical and motivating examples in the study of free boundary problems. A milestone in this context is the classical work of Caffarelli, in which he established for the first time the regularity of free boundaries in the obstacle problem, outside a certain set of singular points. This is one of the main results for which he got the Wolf Prize in 2012 and the Shaw Prize in 2018.
The goal of these notes is to introduce the obstacle problem, prove some of the main known results in this context, and give an overview of more recent research on this topic.

1 Introduction

The most basic mathematical question in PDEs is that of regularity:

Are all solutions to a given PDE smooth, or may they have singularities?

In some cases, all solutions are smooth and no singularities appear (as in Hilbert's XIXth problem [12, 30, 31]). However, in many other cases singularities do appear, and then the main issue becomes the structure of the singular set.

In these notes we will study a special type of elliptic PDE: a *free boundary problem*. In this kind of problems we are not only interested in the regularity of

Xavier Ros-Oton
Universität Zürich, Institut für Mathematik, Winterthurerstrasse, 8057 Zürich, Switzerland, e-mail: xavier.ros-oton@math.uzh.ch

a solution u, but also in the study of an a priori unknown interphase Γ (the free boundary).

As explained later, there is a wide variety of problems in physics, industry, biology, finance, and other areas which can be described by PDEs that exhibit free boundaries. Many of such problems can be written as variational inequalities, for which the solution is obtained by minimizing a constrained energy functional. And the most important and canonical example is the classical *obstacle problem*.

The obstacle problem

Given a smooth function φ, the obstacle problem is the following:

$$\text{minimize} \qquad \frac{1}{2}\int_\Omega |\nabla v|^2 dx \qquad \text{among all functions } v \geq \varphi. \tag{1}$$

Here, the minimization is subject to boundary conditions $v|_{\partial\Omega} = g$.

The interpretation of such problem is clear: One looks for the least energy function v, but the set of admissible functions consists only of functions that are above a certain "obstacle" φ.

In 2D, one can think of the solution v as a "membrane" which is elastic and is constrained to be above φ (see Figure 1).

The Euler–Lagrange equation of the minimization problem is the following:

$$\begin{cases} v \geq \varphi \text{ in } \Omega \\ \Delta v \leq 0 \text{ in } \Omega \\ \Delta v = 0 \text{ in the set } \{v > \varphi\}, \end{cases} \tag{2}$$

together with the boundary conditions $v|_{\partial\Omega} = g$.

Indeed, notice that if we denote $\mathcal{E}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx$, then we will have

$$\mathcal{E}(v + \varepsilon\eta) \geq \mathcal{E}(v) \quad \text{for every } \varepsilon \geq 0 \text{ and } \eta \geq 0,\ \eta \in C_c^\infty(\Omega),$$

which yields $\Delta v \leq 0$ in Ω. That is, we can perturb v with *nonnegative* functions $(\varepsilon\eta)$ and we always get admissible functions $(v + \varepsilon\eta)$. However, due to the constraint $v \geq \varphi$, we cannot perturb v with negative functions in all of Ω, but only in the set $\{v > \varphi\}$. This is why we get $\Delta v \leq 0$ *everywhere* in Ω, but $\Delta v = 0$ *only* in $\{v > \varphi\}$. (We will show later that any minimizer v of (1) is continuous, so that $\{v > \varphi\}$ is open.)

Alternatively, we may consider $u := v - \varphi$, and the problem is equivalent to

$$\begin{cases} u \geq 0 \text{ in } \Omega \\ \Delta u \leq f \text{ in } \Omega \\ \Delta u = f \text{ in the set } \{u > 0\}, \end{cases} \tag{3}$$

where $f := -\Delta\varphi$.

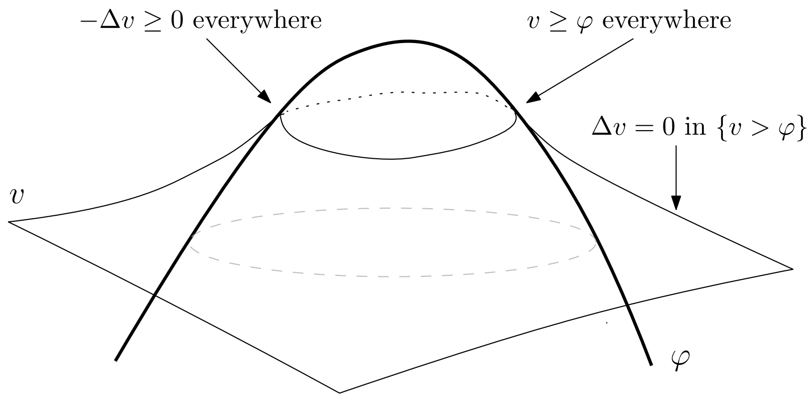

Fig. 1 The function v minimizers the Dirichlet energy among all functions with the same boundary values situated above the obstacle.

Such solution u can be obtained as follows:

$$\text{minimize} \quad \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + fu \right\} dx \quad \text{among all functions } u \geq 0. \tag{4}$$

In other words, we can make the *obstacle* just *zero*, by adding a *right-hand side* f. Here, the minimization is subject to the boundary conditions $u|_{\partial\Omega} = \tilde{g}$, with $\tilde{g} := g - \varphi$.

On the Euler–Lagrange equations

As said above, the Euler–Lagrange equations of the minimization problem (1) are:

(i) $v \geq \varphi$ in Ω (v is *above* the *obstacle*).
(ii) $\Delta v \leq 0$ in Ω (v is a *supersolution*).
(iii) $\Delta v = 0$ in $\{v > \varphi\}$ (v is *harmonic* where it *does not touch* the obstacle).

These are inequalities, rather than a single PDE. Alternatively, one can write also the Euler–Lagrange equations in the following way

$$\min\{-\Delta v, v - \varphi\} = 0 \quad \text{in} \quad \Omega. \tag{5}$$

Of course, the same can be done for the equivalent problem (3). In that case, moreover, the minimization problem (4) is equivalent to

$$\text{minimize} \qquad \int_\Omega \left\{ \frac{1}{2}|\nabla u|^2 + fu^+ \right\} dx, \tag{6}$$

where $u^+ = \max\{u, 0\}$. In this way, we can see the problem not as a constrained minimization but as a minimization problem with a non-smooth term u^+ in the functional. The Euler–Lagrange equation for this functional is then

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in} \quad \Omega. \tag{7}$$

(Here, χ_A denotes the characteristic function of a set $A \subset \mathbb{R}^n$.) We will show this in detail later.

The free boundary

Let us take a closer look at the obstacle problem (3).

One of the most important features of such problem is that it has *two* unknowns: the *solution* u, and the *contact set* $\{u = 0\}$. In other words, there are two regions in Ω: one in which $u = 0$; and one in which $\Delta u = f$.

These regions are characterized by the minimization problem (4). Moreover, if we denote

$$\Gamma := \partial\{u > 0\} \cap \Omega,$$

then this is called the *free boundary*, see Figure 2.

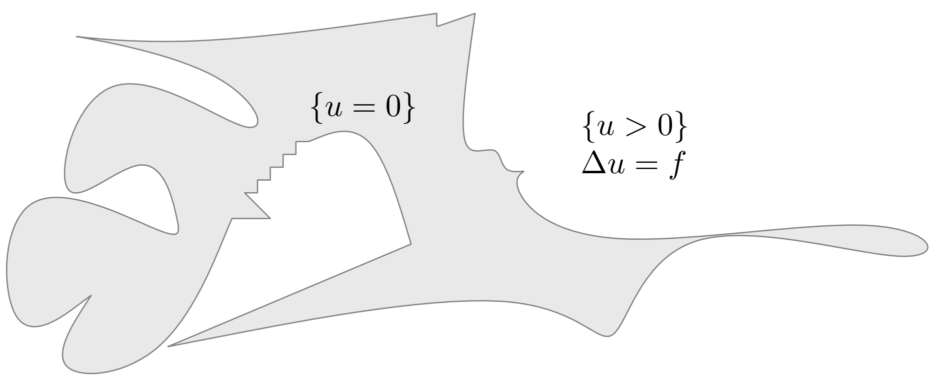

Fig. 2 The free boundary could, a priori, be very irregular.

The obstacle problem is a *free boundary problem*, as it involves an *unknown interface* Γ as part of the problem.

Moreover, if we assume Γ to be smooth, then it is easy to see that the fact that u is a nonnegative supersolution must imply $\nabla u = 0$ on Γ, that is, we will have that $u \geq 0$ solves

$$\begin{cases} \Delta u = f & \text{in } \{u > 0\} \\ \quad u = 0 & \text{on } \Gamma \\ \nabla u = 0 & \text{on } \Gamma. \end{cases} \tag{8}$$

This is an alternative way to write the Euler–Lagrange equation of the problem. In this way, the interface Γ appears clearly, and we see that we have *both* Dirichlet *and* Neumann conditions on Γ.

This would usually be an over-determined problem (too many boundary conditions on Γ), but since Γ is also free, it turns out that the problem has a unique solution (where Γ is part of the solution, of course).

Acknowledgements

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2 Some motivations and applications

Let us briefly comment on some of the main motivations and applications in the study of the obstacle problem. We refer to the books [18, 16, 25, 33, 21, 32], for more details and further applications of obstacle-type problems.

Fluid filtration

The so-called Dam problem aims to describe the filtration of water inside a porous dam. One considers a dam separating two reservoirs of water at different heights, made of a porous medium (permeable to water). Then there is some transfer of water across the dam, and the interior of the dam has a wet part, where water flows, and a dry part. In this setting, an integral of the pressure (with respect to the height of the column of water at each point) solves the obstacle problem, and the free boundary corresponds precisely to the interphase separating the wet and dry parts of the dam.

Phase transitions

The Stefan problem, dating back to the 19th century, is one of the most classical and important free boundary problems. It describes the temperature of a homogeneous medium undergoing a phase change, typically a body of ice at zero degrees submerged in water.

In this context, it turns out that the integral of the temperature $\theta(x,t)$, namely $u(x,t) := \int_0^t \theta$, solves the parabolic version of the obstacle problem,

$$\begin{aligned} u_t - \Delta u &= \chi_{\{u>0\}} \quad \text{in} \quad \Omega \times (0,T) \subset \mathbb{R}^3 \times \mathbb{R}, \\ \partial_t u &\geq 0, \\ u &\geq 0. \end{aligned}$$

The moving interphase separating the solid and liquid is exactly the free boundary $\partial\{u > 0\}$.

Hele-Shaw flow

This 2D model, dating back to 1898, describes a fluid flow between two flat parallel plates separated by a very thin gap. Various problems in fluid mechanics can be approximated to Hele-Shaw flows, and that is why understanding these flows is important.

A Hele-Shaw cell is an experimental device in which a viscous fluid is sandwiched in a narrow gap between two parallel plates. In certain regions, the gap is filled with fluid while in others the gap is filled with air. When liquid is injected inside the device though some sinks (e.g. though a small hole on the top plate) the region filled with liquid grows. In this context, an integral of the pressure solves, for each fixed time t, the obstacle problem. In a similar way to the Dam problem, the free boundary corresponds to the interface between the fluid and the air regions.

Optimal stopping, finance

In probability and finance, the obstacle problem appears when considering optimal stopping problems for stochastic processes.

Indeed, consider a random walk (Brownian motion) inside a domain $\Omega \subset \mathbb{R}^n$, and a payoff function φ defined on the same domain. We can stop the random walk at any moment, and we get the payoff at that position. We want to maximize the expected payoff (by choosing appropriately the stopping strategy). Then, it turns out that the highest expected payoff $v(x)$ starting at a given position x satisfies the obstacle problem (2), where the contact set $\{v = \varphi\}$ is the region where we should immediately stop the random walk and get the payoff, while $\{v > \varphi\}$ is the region where we should wait.

Interacting particle systems

Large systems of interacting particles arise in physical, biological, or material sciences.

In some some models the particles attract each other when they are far, and experience a repulsive force when they are close. In other related models in statistical mechanics, the particles (e.g. electrons) repel with a Coulomb force and one wants to understand their behavior in presence of some external field that confines them.

In this kind of models, a natural and interesting question is to determine the "equilibrium configurations". For instance, in Coulomb systems the charges accumulate in some region with a well defined boundary. Interestingly, these problems are equivalent to the obstacle problem — namely, the electric potential $u = u(x)$ generated by the charges solves such problem — and the contact set $\{u = 0\}$ corresponds to the region in which the particles concentrate.

3 Basic properties of solutions

We proceed now to study the basic properties of solutions $u \geq 0$ to the obstacle problem (4) or (6).

Existence of solutions

Since problem (4) is a minimization of a convex functional, the existence and uniqueness of solutions is standard.

Proposition 1 (Existence and uniqueness) *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $g : \partial\Omega \to \mathbb{R}$ be such that*

$$C = \left\{ u \in H^1(\Omega) : u \geq 0 \text{ in } \Omega,\ u|_{\partial\Omega} = g \right\} \neq \varnothing.$$

Then, for any $f \in L^2(\Omega)$ there exists a unique minimizer of

$$\frac{1}{2}\int_\Omega |\nabla u|^2 dx + \int_\Omega f u$$

among all functions u satisfying $u \geq 0$ in Ω and $u|_{\partial\Omega} = g$.

The proof is left as an exercise to the reader.

Furthermore, we have the following equivalence. (Recall that we denote $u^+ = \max\{u,0\}$, and $u^- = \max\{-u,0\}$, so that $u = u^+ - u^-$.)

Proposition 2 *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $g : \partial\Omega \to \mathbb{R}$ be such that*

$$C = \left\{u \in H^1(\Omega) : u \geq 0 \textit{ in } \Omega,\ u|_{\partial\Omega} = g\right\} \neq \varnothing.$$

Then, the following are equivalent.

(i) u minimizes $\frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu$ among all functions satisfying $u \geq 0$ in Ω and $u|_{\partial\Omega} = g$.
(ii) u minimizes $\frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+$ among all functions satisfying $u|_{\partial\Omega} = g$.

Proof The two functionals coincide whenever $u \geq 0$. Thus, the only key point is to prove that the minimizer in (ii) must be nonnegative, i.e., $u = u^+$. (Notice that since $C \neq \varnothing$ then $g \geq 0$ on $\partial\Omega$.) To show this, recall that the positive part of any H^1 function is still in H^1, and moreover $|\nabla u|^2 = |\nabla u^+|^2 + |\nabla u^-|^2$. Thus, we have that

$$\frac{1}{2}\int_\Omega |\nabla u^+|^2 + \int_\Omega fu^+ \leq \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+,$$

with strict inequality unless $u = u^+$. This means that any minimizer u of the functional in (ii) must be nonnegative, and thus we are done. □

Let us next prove that any minimizer of (4) is actually a solution to (9) below.

We recall that we always assuming that obstacles are as smooth as necessary, $\varphi \in C^\infty(\Omega)$, and therefore we assume here that $f \in C^\infty(\Omega)$ as well.

Proposition 3 *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, $f \in C^\infty(\Omega)$, and u be any minimizer of* (4) *subject to the boundary conditions $u|_{\partial\Omega} = g$.*

Then, u solves

$$\Delta u = f\chi_{\{u>0\}} \quad \textit{in} \quad \Omega \tag{9}$$

in the weak sense.

Proof Notice that, by Proposition 2, u is actually a minimizer of

$$\mathcal{E}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \int_\Omega fu^+$$

subject to the boundary conditions $u|_{\partial\Omega} = g$.

Thus, for any $\eta \in H_0^1(\Omega)$, we have

$$\mathcal{E}(u + \varepsilon\eta) \geq \mathcal{E}(u),$$

where

$$\mathcal{E}(u + \varepsilon\eta) = \mathcal{E}(u) + \varepsilon\int_\Omega \nabla u \cdot \nabla\eta + \frac{\varepsilon^2}{2}\int_\Omega |\nabla\eta|^2 + \int_\Omega f(u + \varepsilon\eta)^+.$$

Now, we want to take the derivative in ε at $\varepsilon = 0$. However, notice that

$$\lim_{\varepsilon\to 0^+} \frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon} = \begin{cases} \eta & \text{in} \quad \{u>0\} \\ \eta^+ & \text{in} \quad \{u=0\}. \end{cases}$$

and

$$\lim_{\varepsilon\to 0^-} \frac{(u+\varepsilon\eta)^+ - u^+}{\varepsilon} = \begin{cases} \eta & \text{in} \quad \{u>0\} \\ \eta^- & \text{in} \quad \{u=0\}. \end{cases}$$

Thus, taking first $\eta \le 0$, and using the dominated convergence theorem, we get

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^+} \mathcal{E}(u+\varepsilon\eta) = \int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\chi_{\{u>0\}}\eta$$

for all $\eta \in H_0^1(\Omega)$ with $\eta \le 0$ in Ω. Then, taking $\eta \ge 0$ we get

$$0 = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0^-} \mathcal{E}(u+\varepsilon\eta) = \int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\chi_{\{u>0\}}\eta$$

for all $\eta \in H_0^1(\Omega)$ with $\eta \ge 0$ in Ω.

Thus, we have proved that the identity

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\chi_{\{u>0\}}\eta = 0$$

holds for every $\eta \ge 0$, and also for every $\eta \le 0$. But then for any $\eta \in H_0^1(\Omega)$ we can use that $\eta = \eta^+ - \eta^-$ in Ω, and $\nabla\eta = \nabla\eta^+ - \nabla\eta^-$ a.e. in Ω, to deduce that

$$\int_\Omega \nabla u \cdot \nabla \eta + \int_\Omega f\chi_{\{u>0\}}\eta = 0 \qquad \text{for all } \eta \in H_0^1(\Omega).$$

Hence, $\Delta u = f\chi_{\{u>0\}}$ in Ω, as wanted. □

As a consequence, we find:

Corollary 1 *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, $f \in C^\infty(\Omega)$, and u be any minimizer of* (4) *subject to the boundary conditions $u|_{\partial\Omega} = g$.*

Then, u is $C^{1,\alpha}$ inside Ω, for every $\alpha \in (0,1)$.

Proof By Proposition 3, u solves

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in} \quad \Omega.$$

Since $f\chi_{\{u>0\}} \in L^\infty(\Omega)$, then by standard regularity estimates we deduce that $u \in C^{1,1-\varepsilon}$ for every $\varepsilon > 0$. □

Optimal regularity of solutions

Thanks to the previous results, we know that any minimizer of (4) solves (9) and is $C^{1,\alpha}$. From now on, we will localize the problem and study it in a ball:

$$\begin{aligned} u &\geq 0 && \text{in } B_1 \\ \Delta u &= f\chi_{\{u>0\}} && \text{in } B_1. \end{aligned} \tag{10}$$

Our next goal is to answer the following question:

Question: *What is the optimal regularity of solutions?*

First, a few important considerations. Notice that in the set $\{u > 0\}$ we have $\Delta u = f$, while in the interior of $\{u = 0\}$ we have $\Delta u = 0$ (since $u \equiv 0$ there).

Thus, since f is in general not zero, then Δu is *discontinuous* across the free boundary $\partial\{u > 0\}$ in general. In particular, $u \notin C^2$.

We will now prove that any minimizer of (4) is actually $C^{1,1}$, which gives the:

Answer: $u \in C^{1,1}$ *(second derivatives are bounded but not continuous)*

The precise statement and proof are given next.

Theorem 1 (Optimal regularity)
Let $f \in C^\infty(B_1)$, and u be any solution to (10)*. Then, u is $C^{1,1}$ inside $B_{1/2}$, with the estimate*

$$\|u\|_{C^{1,1}(B_{1/2})} \leq C\big(\|u\|_{L^\infty(B_1)} + \|f\|_{\mathrm{Lip}(B_1)}\big).$$

The constant C depends only on n.

To prove this, the main step is the following.

Lemma 1 *Let u be any solution to* (10)*. Let $x_\circ \in \overline{B_{1/2}}$ be any point on $\{u = 0\}$. Then, for any $r \in (0, \frac{1}{4})$ we have*

$$0 \leq \sup_{B_r(x_\circ)} u \leq Cr^2,$$

with C depending only on n and $\|f\|_{L^\infty(B_1)}$.

Proof We have that $\Delta u = f\chi_{\{u>0\}}$ in B_1, with $f\chi_{\{u>0\}} \in L^\infty(B_1)$. Thus, since $u \geq 0$, we can use the Harnack inequality for the equation $\Delta u = f\chi_{\{u>0\}}$ in $B_{2r}(x_\circ)$, to find

$$\sup_{B_r(x_\circ)} u \leq C\left(\inf_{B_r(x_\circ)} u + r^2\|f\chi_{\{u>0\}}\|_{L^\infty(B_{2r}(x_\circ))}\right).$$

Since $u \geq 0$ and $u(x_\circ) = 0$, this yields $\sup_{B_r(x_\circ)} u \leq C\|f\|_{L^\infty(B_1)} r^2$, as wanted. □

We have proved that:

At every free boundary point $x_\circ$, u grows (at most) quadratically.

As shown next, this easily implies the $C^{1,1}$ regularity.

Proof (Proof of Theorem 1) Dividing u by a constant if necessary, we may assume that $\|u\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(B_1)} \leq 1$, where $\alpha \in (0, 1)$ is fixed.

We already know that $u \in C^\infty$ in the set $\{u > 0\}$ (since $\Delta u = f \in C^\infty$), and also inside the set $\{u = 0\}$ (since $u = 0$ there). Moreover, on the interface $\Gamma = \partial\{u > 0\}$ we have proved the quadratic growth $\sup_{B_r(x_\circ)} u \le Cr^2$. Let us prove that this yields the $C^{1,1}$ bound we want.

Let $x_1 \in \{u > 0\} \cap B_{1/2}$, and let $x_\circ \in \Gamma$ be the closest free boundary point. Denote $\rho = |x_1 - x_\circ|$. Then, we have $\Delta u = f$ in $B_\rho(x_1)$.

By Schauder estimates, we find

$$\|D^2 u\|_{L^\infty(B_{\rho/2}(x_1))} \le C\left(\frac{1}{\rho^2}\|u\|_{L^\infty(B_\rho(x_1))} + \|f\|_{C^{0,\alpha}(B_1)}\right).$$

But by the growth proved in the previous Lemma, we have $\|u\|_{L^\infty(B_\rho(x_1))} \le C\rho^2$, which yields

$$\|D^2 u\|_{L^\infty(B_{\rho/2}(x_1))} \le C.$$

In particular,

$$|D^2 u(x_1)| \le C.$$

Since we can do this for each $x_1 \in \{u > 0\} \cap B_{1/2}$, it follows that $\|u\|_{C^{1,1}(B_{1/2})} \le C$, as wanted. □

The overall strategy of the proof of optimal regularity is summarized in Figure 3.

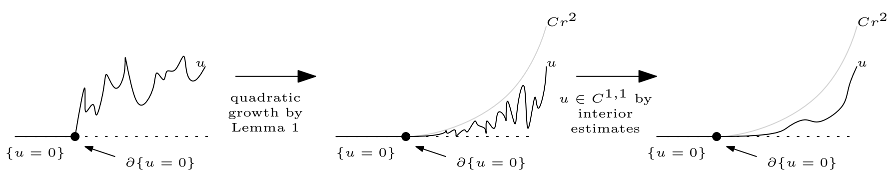

Fig. 3 Strategy of the proof of Theorem 1.

Nondegeneracy

We now want to prove that, at all free boundary points, u grows *at least* quadratically (we already know *at most* quadratically).

That is, we want

$$0 < cr^2 \le \sup_{B_r(x_\circ)} u \le Cr^2 \qquad (11)$$

for all free boundary points $x_\circ \in \partial\{u > 0\}$.

This property is essential in order to study the free boundary later.

For this, we need the following:

Assumption: *The right hand side f satisfies*

$$f \geq c_\circ > 0$$

in the ball B_1.

(Actually, it is common to simply assume $f \equiv 1$, since this is the right hand side that arises naturally in many models.)

Proposition 4 (Nondegeneracy)

Let u be any solution to (10). *Assume that $f \geq c_\circ > 0$ in B_1. Then, for every free boundary point $x_\circ \in \partial\{u > 0\} \cap B_{1/2}$, we have*

$$0 < cr^2 \leq \sup_{B_r(x_\circ)} u \leq Cr^2 \qquad \text{for all } r \in (0, \tfrac{1}{2}),$$

with a constant $c > 0$ depending only on n and $c_\circ$.

Proof Let $x_1 \in \{u > 0\}$ be any point close to $x_\circ$ (we will then let $x_1 \to x_\circ$ at the end of the proof).

Consider the function

$$w(x) := u(x) - \frac{c_\circ}{2n}|x - x_1|^2.$$

Then, in $\{u > 0\}$ we have

$$\Delta w = \Delta u - c_\circ = f - c_\circ \geq 0$$

and hence $-\Delta w \leq 0$ in $\{u > 0\} \cap B_r(x_1)$. Moreover, $w(x_1) > 0$.

By the maximum principle, w attains a positive maximum on $\partial\big(\{u > 0\} \cap B_r(x_1)\big)$. But on the free boundary $\partial\{u > 0\}$ we clearly have $w < 0$. Therefore, there is a point on $\partial B_r(x_1)$ at which $w > 0$. In other words,

$$0 < \sup_{\partial B_r(x_1)} w = \sup_{\partial B_r(x_1)} u - \frac{c_\circ}{2n} r^2.$$

Letting now $x_1 \to x_\circ$, we find $\sup_{\partial B_r(x_\circ)} u \geq cr^2 > 0$, as desired. □

Summary of basic properties

Let u be any solution to the obstacle problem

$$\begin{aligned} u &\geq 0 && \text{in } B_1, \\ \Delta u &= f\chi_{\{u>0\}} && \text{in } B_1. \end{aligned}$$

Then, we have:

- Optimal regularity: $\quad \|u\|_{C^{1,1}(B_{1/2})} \leq C\left(\|u\|_{L^\infty(B_1)} + \|f\|_{\mathrm{Lip}(B_1)}\right)$

• <u>Nondegeneracy</u>: If $f \geq c_\circ > 0$, then

$$0 < cr^2 \leq \sup_{B_r(x_\circ)} u \leq Cr^2 \qquad \text{for all } r \in (0, \tfrac{1}{2})$$

at all free boundary points $x_\circ \in \partial\{u > 0\} \cap B_{1/2}$.

Using these properties, we can now start the study of the free boundary.

4 Regularity of free boundaries: an overview

From now on, we consider any solution to

$$\begin{aligned} &u \in C^{1,1}(B_1), \\ &u \geq 0 \quad \text{in } B_1, \\ &\Delta u = f \quad \text{in } \{u > 0\}, \end{aligned} \tag{12}$$

(see Figure 4) with

$$f \geq c_\circ > 0 \qquad \text{and} \qquad f \in C^\infty. \tag{13}$$

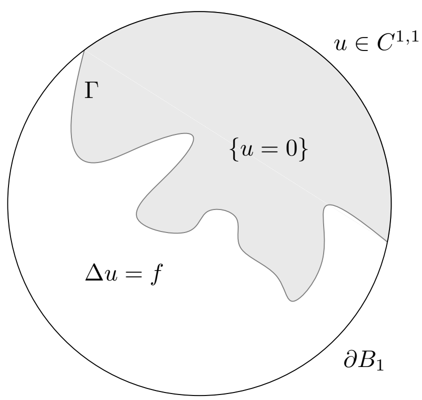

Fig. 4 A solution to the obstacle problem in B_1.

Notice that on the interface

$$\Gamma = \partial\{u > 0\} \cap B_1$$

we have that

$$\begin{aligned} u = 0 &\quad \text{on } \Gamma, \\ \nabla u = 0 &\quad \text{on } \Gamma. \end{aligned}$$

The central mathematical challenge in the obstacle problem is to

Understand the geometry/regularity of the free boundary Γ.

Notice that, even if we already know the optimal regularity of u (it is $C^{1,1}$), we know nothing about the free boundary Γ. A priori Γ could be a very irregular object, even a fractal set with infinite perimeter.

As we will see, under the natural assumption $f \geq c_\circ > 0$, it turns out that free boundaries are always smooth, possibly outside a certain set of singular points. In fact, in our proofs we will assume for simplicity that $f \equiv 1$ (or constant). We do that in order to avoid x-dependence and the technicalities associated to it, which gives cleaner proofs. In this way, the main ideas behind the regularity of free boundaries are exposed.

Regularity of free boundaries: main results

Assume from now on that u solves (12)-(13). Then, the main known results on the free boundary $\Gamma = \partial\{u > 0\}$ can be summarized as follows:

• At every free boundary point $x_\circ \in \Gamma$, we have

$$0 < cr^2 \leq \sup_{B_r(x_\circ)} u \leq Cr^2 \qquad \forall r \in (0, \tfrac{1}{2})$$

• The free boundary Γ splits into *regular points* and *singular points*.

• The set of *regular points* is an open subset of the free boundary, and Γ is C^∞ near these points.

• *Singular points* are those at which the contact set $\{u = 0\}$ has *density zero*, and these points (if any) are contained in an $(n-1)$-dimensional C^1 manifold.

Summarizing, *the free boundary is smooth, possibly outside a certain set of singular points*. See Figure 5.

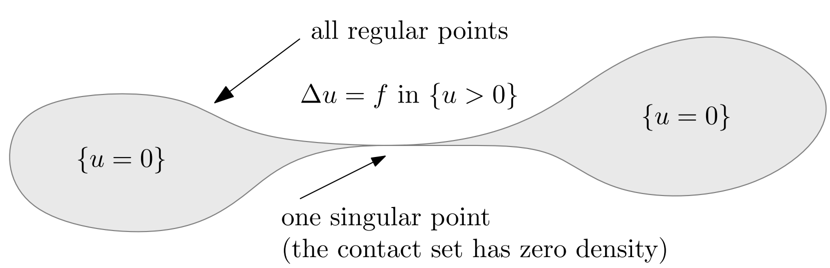

Fig. 5 Singular points are those where the contact set has zero density.

So far, we have not even proved that Γ has finite perimeter, or anything at all about Γ. Our goal will be to prove that Γ *is* C^∞ *near regular points*. This is the main and most important result in the obstacle problem. It was proved by Caffarelli in 1977, and it is one of the major results for which he received the Wolf Prize in 2012 and the Shaw Prize in 2018.

Overview of the strategy

To prove these regularity results for the free boundary, one considers *blow-ups*. Namely, given any free boundary point $x_\circ$ for a solution u of (12)-(13), one takes the rescalings

$$u_r(x) := \frac{u(x_\circ + rx)}{r^2},$$

with $r > 0$ small. This is like "zooming in" at a free boundary point.

The factor r^{-2} is chosen so that

$$\|u_r\|_{L^\infty(B_1)} \approx 1$$

as $r \to 0$; recall that $0 < cr^2 \le \sup_{B_r(x_\circ)} u \le Cr^2$.

Then, by $C^{1,1}$ estimates, we will prove that a subsequence of u_r converges to a function u_0 locally uniformly in $\mathbb{R}^n$ as $r \to 0$. Such function u_0 is called a *blow-up of u at $x_\circ$*.

Any blow-up u_0 is a *global* solution to the obstacle problem, with $f \equiv 1$ (or with $f \equiv \mathrm{ctt} > 0$).

Then, the main issue is to *classify blow-ups*: that is, to show that

either $\quad u_0(x) = \frac{1}{2}(x \cdot e)_+^2 \quad$ (this happens at regular points)

or $\quad u_0(x) = \frac{1}{2}x^T A x \quad$ (this happens at singular points).

Here, $e \in \mathbb{S}^{n-1}$ is a unit vector, and $A \ge 0$ is a positive semi-definite matrix satisfying $\mathrm{tr} A = 1$. Notice that the contact set $\{u_0 = 0\}$ becomes a half-space in case of regular points, while it has zero measure in case of singular points; see Figure 6.

Once this is done, one has to "transfer" the information from the blow-up u_0 to the original solution u. Namely, one shows that, in fact, the free boundary is $C^{1,\alpha}$ near regular points (for some small $\alpha > 0$).

Finally, once we know that the free boundary is $C^{1,\alpha}$, then we will "bootstrap" the regularity to C^∞, by using fine estimates for harmonic functions in $C^{k,\alpha}$ domains.

Classifying blow-ups is not easy. Generally speaking, classifying blow-ups is of similar difficulty to proving regularity estimates.

Thus, how can we classify blow-ups? Do we get any extra information on u_0 that we did not have for u? (Otherwise it seems hopeless!)

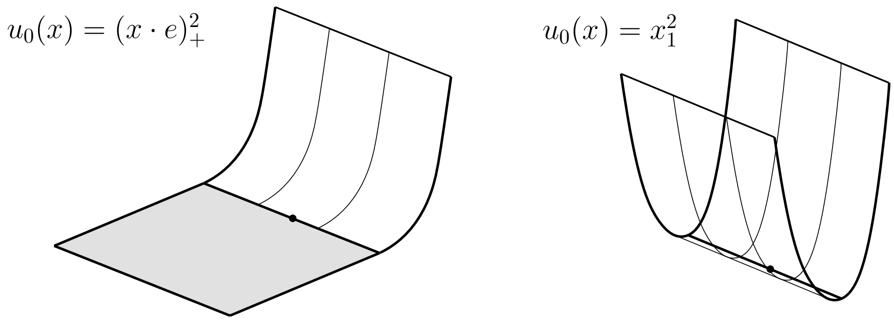

Fig. 6 Possible blow-ups of the solution to the obstacle problem at free boundary points.

The answer is *yes*: Convexity. We will prove that all blow-ups are always *convex*. This is a huge improvement, since this yields that the contact set $\{u_0 = 0\}$ is also convex. Furthermore, we will show that blow-ups are also *homogeneous*.

So, before the blow-up we had no information on the set $\{u = 0\}$, but after the blow-up we get that $\{u_0 = 0\}$ *is a convex cone*. Thanks to this we will be able to classify blow-ups, and thus to prove the regularity of the free boundary.

The main steps in the proof of the regularity of the free boundary will be the following:

1. $0 < cr^2 \leq \sup_{B_r(x_\circ)} u \leq Cr^2$
2. Blow-ups u_0 are *convex* and *homogeneous*.
3. If the contact set has *positive density* at $x_\circ$, then $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$.
4. Deduce that the free boundary is $C^{1,\alpha}$ near $x_\circ$.
5. Deduce that the free boundary is C^∞ near $x_\circ$.

The proofs that we will present here are a modified version of the original ones due to Caffarelli (see [7]), together with some extra tools due to Weiss (see [40]). We refer to [7], [32], and [40], for different proofs of the classification of blow-ups and/or of the regularity of free boundaries.

5 Classification of blow-ups

The aim of this Section is to classify all possible blow-ups u_0.

Convexity of blow-ups

The first important property about blow-ups is that they are convex. More precisely, for the original solution u in B_1, the closer we look to a free boundary point $x_\circ$, the closer is the solution to being convex.

Recall that, for simplicity, from now on we will assume that $f \equiv 1$ in B_1. This is only to avoid x-dependence in the equation.

Therefore, from now on we consider a solution u satisfying (see Figure 7):

$$\begin{aligned} & u \in C^{1,1}(B_1) \\ & u \geq 0 \quad \text{in } B_1 \\ & \Delta u = 1 \quad \text{in } \{u > 0\} \\ & 0 \text{ is a free boundary point.} \end{aligned} \tag{14}$$

We will prove all the results around the origin (without loss of generality).

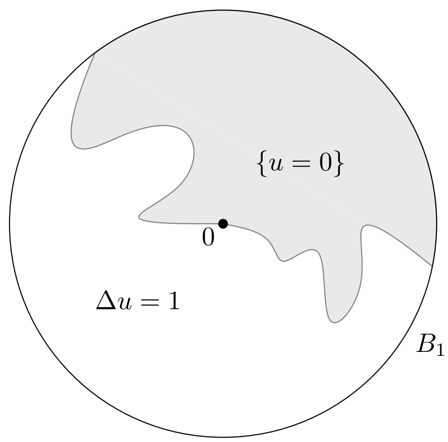

Fig. 7 A solution u to the obstacle problem with $f \equiv 1$.

The convexity of blow-ups is given by the following.

Theorem 2 *Let $u_0 \in C^{1,1}$ be any global solution to*

$$\begin{aligned} & u_0 \geq 0 \quad \textit{in } \mathbb{R}^n \\ & \Delta u_0 = 1 \quad \textit{in } \{u_0 > 0\} \\ & 0 \textit{ is a free boundary point.} \end{aligned}$$

Then, u_0 is convex.

We skip the proof of such result, and we refer to [7], [18] or [32] for more details.

Homogeneity of blow-ups

We will now prove that blow-ups are homogeneous. This is not essential in the proof of the regularity of the free boundary (see [7]), but it actually simplifies it. We will show that, for the original solution u in B_1, the closer we look at a free boundary point $x_\circ$, the closer is the solution to being homogeneous.

Proposition 5 (Homogeneity of blow-ups)

Let u be any solution to (14). *Then, any blow-up of u at* 0 *is homogeneous of degree* 2.

It is important to remark that not all global solutions to the obstacle problem in $\mathbb{R}^n$ are homogeneous. There exist global solutions u_0 that are convex, $C^{1,1}$, and whose contact set $\{u_0 = 0\}$ is an ellipsoid, for example. However, thanks to the previous result, we find that such non-homogeneous solutions cannot appear as blow-ups, i.e., that all blow-ups must be homogeneous.

To prove this, we will need the following monotonicity formula due to Weiss.

Theorem 3 (Weiss' monotonicity formula)

Let u be any solution to (14). *Then, the quantity*

$$W_u(r) := \frac{1}{r^{n+2}} \int_{B_r} \left\{ \tfrac{1}{2}|\nabla u|^2 + u \right\} - \frac{1}{r^{n+3}} \int_{\partial B_r} u^2 \tag{15}$$

is monotone in r, i.e.,

$$\frac{d}{dr} W_u(r) := \frac{1}{r^{n+3}} \int_{\partial B_r} (x \cdot \nabla u - 2u)^2 dx \geq 0$$

for $r \in (0,1)$.

Proof Let $u_r(x) = r^{-2}u(rx)$, and observe that

$$W_u(r) = \int_{B_1} \left\{ \tfrac{1}{2}|\nabla u_r|^2 + u_r \right\} - \int_{\partial B_1} u_r^2.$$

Using this, together with

$$\frac{d}{dr} u_r = \frac{1}{r} \left\{ x \cdot \nabla u_r - 2u_r \right\} \tag{16}$$

and

$$\frac{d}{dr}(\nabla u_r) = \nabla \frac{d}{dr} u_r,$$

we find

$$\frac{d}{dr} W_u(r) = \int_{B_1} \left\{ \nabla u_r \cdot \nabla \frac{d}{dr} u_r + \frac{d}{dr} u_r \right\} - 2 \int_{\partial B_1} u_r \frac{d}{dr} u_r.$$

Now, integrating by parts we get

$$\int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r = -\int_{B_1} \Delta u_r \frac{d}{dr} u_r + \int_{\partial B_1} \partial_\nu (u_r) \frac{d}{dr} u_r .$$

Since $\Delta u_r = 1$ in $\{u_r > 0\}$ and $\frac{d}{dr} u_r = 0$ in $\{u_r = 0\}$, then

$$\int_{B_1} \nabla u_r \cdot \nabla \frac{d}{dr} u_r = -\int_{B_1} \frac{d}{dr} u_r + \int_{\partial B_1} \partial_\nu (u_r) \frac{d}{dr} u_r .$$

Thus, we deduce

$$\frac{d}{dr} W_u(r) = \int_{\partial B_1} \partial_\nu (u_r) \frac{d}{dr} u_r - 2 \int_{\partial B_1} u_r \frac{d}{dr} u_r .$$

Using that on ∂B_1 we have $\partial_\nu = x \cdot \nabla$, combined with (16), yields

$$\frac{d}{dr} W_u(r) = \frac{1}{r} \int_{\partial B_1} (x \cdot \nabla u_r - 2u_r)^2 ,$$

which gives the desired result. □

We now give the:

Proof (Proof of Proposition 5) Let $u_r(x) = r^{-2} u(rx)$, and notice that we have the scaling property

$$W_{u_r}(\rho) = W_u(\rho r),$$

for any $r, \rho > 0$.

If u_0 is any blow-up of u at 0 then there is a sequence $r_j \to 0$ satisfying $u_{r_j} \to u_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Thus, for any $\rho > 0$ we have

$$W_{u_0}(\rho) = \lim_{r_j \to 0} W_{u_{r_j}}(\rho) = \lim_{r_j \to 0} W_u(\rho r_j) = W_u(0^+).$$

Notice that the limit $W_u(0^+) := \lim_{r \to 0} W_u(r)$ exists, by monotonicity of W.

Hence, the function $W_{u_0}(\rho)$ is *constant* in ρ. However, by Theorem 3 this yields that $x \cdot \nabla u_0 - 2u_0 \equiv 0$ in $\mathbb{R}^n$, and therefore u_0 is homogeneous of degree 2. □

Remark 1 Here, we used that a C^1 function u_0 is 2-homogeneous (i.e. $u_0(\lambda x) = \lambda^2 u_0(x)$ for all $\lambda \in \mathbb{R}_+$) if and only if $x \cdot \nabla u_0 \equiv 2u_0$. This is because $\partial_\lambda|_{\lambda=1} \left\{ \lambda^{-2} u(\lambda x) \right\} = x \cdot \nabla u_0 - 2u_0$.

Classification of blow-ups

We next want to classify all possible blow-ups for solutions to the obstacle problem (14). First, we will prove the following.

Proposition 6 *Let u be any solution to* (14)*, and let*

$$u_r(x) := \frac{u(rx)}{r^2}.$$

Then, for any sequence $r_k \to 0$ *there is a subsequence* $r_{k_j} \to 0$ *such that*

$$u_{r_{k_j}} \longrightarrow u_0 \quad \text{in } C^1_{\rm loc}(\mathbb{R}^n)$$

as $k_j \to \infty$*, for some function* u_0 *satisfying*

$$\begin{cases} u_0 \in C^{1,1}_{\rm loc}(\mathbb{R}^n) \\ u_0 \geq 0 \quad \text{in } B_1 \\ \Delta u_0 = 1 \quad \text{in } \{u_0 > 0\} \\ 0 \text{ is a free boundary point} \\ u_0 \text{ is convex} \\ u_0 \text{ is homogeneous of degree } 2. \end{cases} \tag{17}$$

Proof By $C^{1,1}$ regularity of u, and by nondegeneracy, we have that

$$\frac{1}{C} \leq \sup_{B_1} u_r \leq C$$

for some $C > 0$. Moreover, again by $C^{1,1}$ regularity of u, we have

$$\|D^2 u_r\|_{L^\infty(B_{1/r})} \leq C.$$

Since the sequence $\{u_{r_k}\}$, for $r_k \to 0$, is uniformly bounded in $C^{1,1}(K)$ for each compact set $K \subset \mathbb{R}^n$, then there is a subsequence $r_{k_j} \to 0$ such that

$$u_{r_{k_j}} \longrightarrow u_0 \quad \text{in } C^1_{\rm loc}(\mathbb{R}^n)$$

for some $u_0 \in C^{1,1}(K)$. Moreover, such function u_0 satisfies $\|D^2 u_0\|_{L^\infty(K)} \leq C$, with C independent of K, and clearly $u_0 \geq 0$ in K.

The fact that $\Delta u_0 = 1$ in $\{u_0 > 0\} \cap K$ can be checked as follows. For any smooth function $\eta \in C^\infty_c(\{u > 0\} \cap K)$ we will have that, for k_j large enough, $u_{r_{k_j}} > 0$ in the support of η, and thus

$$\int_{\mathbb{R}^n} \nabla u_{r_{k_j}} \cdot \nabla \eta \, dx = \int_{\mathbb{R}^n} \eta \, dx.$$

Since $u_{r_{k_j}} \to u_0$ in $C^1(K)$ then we can take the limit $k_j \to \infty$ to get

$$\int_{\mathbb{R}^n} \nabla u_0 \cdot \nabla \eta \, dx = \int_{\mathbb{R}^n} \eta \, dx.$$

Since this can be done for any $\eta \in C^\infty_c(\{u > 0\} \cap K)$, and for every $K \subset \mathbb{R}^n$, it follows that $\Delta u_0 = 1$ in $\{u_0 > 0\}$.

The fact that 0 is a free boundary point for u_0 follows simply by taking limits to $u_{r_{k_j}}(0) = 0$ and $\|u_{r_{k_j}}\|_{L^\infty(B_\rho)} \approx \rho^2$ for all $\rho \in (0, 1)$. Finally, the convexity and homogeneity of u_0 follow from Theorem 2 and Proposition 5. □

Our next goal is to prove the following.

Theorem 4 (Classification of blow-ups)

Let u be any solution to (14)*, and let u_0 be any blow-up of u at* 0*. Then,*

(a) either

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2$$

for some $e \in \mathbb{S}^{n-1}$.

(b) or

$$u_0(x) = \frac{1}{2}x^T Ax$$

for some matrix $A \geq 0$ with $\operatorname{tr} A = 1$.

It is important to remark here that, a priori, different subsequences could lead to different blow-ups u_0.

In order to establish Theorem 4, we will need the following.

Lemma 2 *Let $\Sigma \subset \mathbb{R}^n$ be any closed convex cone with nonempty interior, and with vertex at the origin. Let $w \in C(\mathbb{R}^n)$ be a function satisfying $\Delta w = 0$ in Σ^c, $w > 0$ in Σ^c, and $w = 0$ in Σ.*

Assume in addition that w is homogeneous of degree 1. Then, Σ must be a half-space.

Proof By convexity of Σ, there exists a half-space $H = \{x \cdot e > 0\}$, with $e \in \mathbb{S}^{n-1}$, such that $H \subset \Sigma^c$.

Let $v(x) = (x \cdot e)_+$, which is harmonic and positive in H, and vanishes in H^c. By Hopf Lemma, we have that $w \geq c_\circ d_\Sigma$ in $\Sigma^c \cap B_1$, where $d_\Sigma(x) = \operatorname{dist}(x, \Sigma)$ and $c_\circ$ is a small positive constant. In particular, since both w and d_Σ are homogeneous of degree 1, we deduce that $w \geq c_\circ d_\Sigma$ in all of Σ^c. Notice that, in order to apply Hopf Lemma, we used that — by convexity of Σ — the domain Σ^c satisfies the interior ball condition.

Thus, since $d_\Sigma \geq d_H = v$, we deduce that $w \geq c_\circ v$, for some $c_\circ > 0$. The idea is now to consider the functions w and cv, and let $c > 0$ increase until the two functions touch at one point, which will give us a contradiction (recall that two harmonic functions cannot touch at an interior point). To do this rigorously, define

$$c_* := \sup\{c > 0 \,:\, w \geq cv \quad \text{in} \quad \Sigma^c\}.$$

Notice that $c_* \geq c_\circ > 0$. Then, we consider the function $w - c_* v \geq 0$. Assume that $w - c_* v$ is not identically zero. Such function is harmonic in H and hence, by the strict maximum principle, $w - c_* v > 0$ in H. Then, using Hopf Lemma in H we deduce that $w - c_* v \geq c_\circ d_H = c_\circ v$, since v is exactly the distance to H^c. But then we get that $w - (c_* + c_\circ)v \geq 0$, a contradiction with the definition of c_*.

Therefore, it must be $w - c_* v \equiv 0$. This means that w is a multiple of v, and therefore $\Sigma = H$, a half-space. □

Remark 2 (Alternative proof) An alternative way to argue in the previous lemma could be the following. Any function w which is harmonic in a cone Σ^c and homogeneous of degree α can be written as a function on the sphere, satisfying $\Delta_{\mathbb{S}^{n-1}} w = \mu w$ on $\mathbb{S}^{n-1} \cap \Sigma^c$ with $\mu = \alpha(n + \alpha - 2)$ — in our case $\alpha = 1$. (Here, $\Delta_{\mathbb{S}^{n-1}}$ denotes the spherical Laplacian, i.e. the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$.) In other words, *homogeneous harmonic functions solve an eigenvalue problem on the sphere*.

Using this, we notice that $w > 0$ in Σ^c and $w = 0$ in Σ imply that w is the *first* eigenfunction of $\mathbb{S}^{n-1} \cap \Sigma^c$, and that the first eigenvalue is $\mu = n - 2$. But, on the other hand, the same happens for the domain $H = \{x \cdot e > 0\}$, since $v(x) = (x \cdot e)_+$ is a positive harmonic function in H. This means that both domains $\mathbb{S}^{n-1} \cap \Sigma^c$ and $\mathbb{S}^{n-1} \cap H$ have the same first eigenvalue μ. But then, by strict monotonicity of the first eigenvalue with respect to domain inclusions, we deduce that $H \subset \Sigma^c \Longrightarrow H = \Sigma^c$, as desired.

We will also need the following.

Lemma 3 *Assume that $\Delta u = 1$ in $\mathbb{R}^n \setminus \partial H$, where $\partial H = \{x_1 = 0\}$ is a hyperplane. If $u \in C^1(\mathbb{R}^n)$, then $\Delta u = 1$ in $\mathbb{R}^n$.*

Proof For any ball $B_R \subset \mathbb{R}^n$, we consider the solution to $\Delta w = 1$ in B_R, $w = u$ on ∂B_R, and define $v = u - w$. Then, we have $\Delta v = 0$ in $B_R \setminus \partial H$, and $v = 0$ on ∂B_R. We want to show that u coincides with w, that is, $v \equiv 0$ in B_R.

For this, notice that since v is bounded then for $\kappa > 0$ large enough we have

$$v(x) \leq \kappa(2R - |x_1|) \quad \text{in} \quad B_R,$$

where $2R - |x_1|$ is positive in B_R and harmonic in $B_R \setminus \{x_1 = 0\}$. Thus, we may consider $\kappa^* := \inf\{\kappa \geq 0 : v(x) \leq \kappa(2R - |x_1|) \quad \text{in} \quad B_R\}$. Assume $\kappa^* > 0$. Since v and $2R - |x_1|$ are continuous in B_R, and $v = 0$ on ∂B_R, then we must have a point $p \in B_R$ at which $v(p) = \kappa^*(2R - |p_1|)$. Moreover, since v is C^1, and the function $2R - |x_1|$ has a wedge on $H = \{x_1 = 0\}$, then we must have $p \in B_R \setminus H$. However, this is not possible, as two harmonic functions cannot touch tangentially at an interior point p. This means that $\kappa^* = 0$, and hence $v \leq 0$ in B_R. Repeating the same argument with $-v$ instead of v, we deduce that $v \equiv 0$ in B_R, and thus the lemma is proved. □

Finally, we will use the following basic property of convex functions, whose proof is left as an exercise to the reader.

Lemma 4 *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that the set $\{u = 0\}$ contains the straight line $\{te' : t \in \mathbb{R}\}$, $e' \in \mathbb{S}^{n-1}$. Then, $u(x + te') = u(x)$ for all $x \in \mathbb{R}^n$ and all $t \in \mathbb{R}$.*

Proof After a rotation, we may assume $e' = e_n$. Then, writing $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have that $u(0, x_n) = 0$ for all $x_n \in \mathbb{R}$, and we want to prove that $u(x', x_n) = u(x', 0)$ for all $x' \in \mathbb{R}^{n-1}$ and all $x_n \in \mathbb{R}$.

Now, by convexity, given x' and x_n, for every $\varepsilon > 0$ and $M \in \mathbb{R}$ we have

$$(1-\varepsilon)u(x', x_n) + \varepsilon u(0, x_n + M) \geq u((1-\varepsilon)x', x_n + \varepsilon M).$$

Since $u(0, x_n + M) = 0$, then choosing $M = \lambda/\varepsilon$ and letting $\varepsilon \to 0$ we deduce that

$$u(x', x_n) \geq u(x', x_n + \lambda).$$

Finally, since this can be done for any $\lambda \in \mathbb{R}$ and $x_n \in \mathbb{R}$, the result follows. □

We finally establish the classification of blow-ups at regular points.

Proof (Proof of Theorem 4) Let u_0 be any blow-up of u at 0. We already proved that u_0 is convex and homogeneous of degree 2. We divide the proof into two cases.

Case 1. Assume that $\{u_0 = 0\}$ has nonempty interior. Then, we have $\{u_0 = 0\} = \Sigma$, a closed convex cone with nonempty interior.

For any direction $\tau \in \mathbb{S}^{n-1}$ such that $-\tau \in \mathring{\Sigma}$, we claim that $\partial_\tau u_0 \geq 0$ in $\mathbb{R}^n$. Indeed, for every $x \in \mathbb{R}^n$ we have that $u_0(x + \tau t)$ is zero for $t \ll -1$, and therefore by convexity of u_0 we get that $\partial_t u_0(x + \tau t)$ is monotone nondecreasing in t, and zero for $t \ll -1$. This means that $\partial_t u_0 \geq 0$, and thus $\partial_\tau u_0 \geq 0$ in $\mathbb{R}^n$, as claimed.

Now, for any such τ, we define $w := \partial_\tau u_0 \geq 0$. Notice that, at least for some $\tau \in \mathbb{S}^{n-1}$ with $-\tau \in \mathring{\Sigma}$, the function w is not identically zero. Moreover, since it is harmonic in Σ^c — recall that $\Delta u_0 = 1$ in Σ^c — then $w > 0$ in Σ^c.

But then, since w is homogeneous of degree 1, we can apply Lemma 2 to deduce that we must necessarily have that Σ is a half-space.

By convexity of u_0 and Lemma 4, this means that u_0 is a 1D function, i.e., $u_0(x) = U(x \cdot e)$ for some $U : \mathbb{R} \to \mathbb{R}$ and some $e \in \mathbb{S}^{n-1}$. Thus, we have that $U \in C^{1,1}$ solves $U''(t) = 1$ for $t > 0$, with $U(t) = 0$ for $t \leq 0$. We deduce that $U(t) = \frac{1}{2}t_+^2$, and therefore $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$.

Case 2. Assume now that $\{u_0 = 0\}$ has empty interior. Then, by convexity, $\{u_0 = 0\}$ is contained in a hyperplane ∂H. Hence, $\Delta u_0 = 1$ in $\mathbb{R}^n \setminus \partial H$, with ∂H being a hyperplane, and $u_0 \in C^{1,1}$. It follows from Lemma 3 that $\Delta u_0 = 1$ in all of $\mathbb{R}^n$. But then all second derivatives of u_0 are harmonic and globally bounded in $\mathbb{R}^n$, so they must be constant. Hence, u_0 is a quadratic polynomial. Finally, since $u_0(0) = 0$, $\nabla u_0(0) = 0$, and $u_0 \geq 0$, then it must be $u_0(x) = \frac{1}{2}x^T Ax$ for some $A \geq 0$, and since $\Delta u_0 = 1$ then $\operatorname{tr} A = 1$. □

6 Regularity of the free boundary

The aim of this Section is to prove Theorem 6 below, i.e., that if u is any solution to (14) satisfying

$$\limsup_{r\to 0} \frac{\big|\{u=0\}\cap B_r\big|}{|B_r|} > 0 \tag{18}$$

(i.e., the contact set has positive density at the origin), then the free boundary $\partial\{u>0\}$ is C^∞ in a neighborhood of the origin.

For this, we will use the classification of blow-ups established in the previous Section.

$C^{1,\alpha}$ regularity of the free boundary

The first step here is to transfer the local information on u given by (18) into a blow-up u_0. More precisely, we next show that

(18) $\Longrightarrow$ The contact set of a blow-up u_0 has nonempty interior.

Lemma 5 *Let u be any solution to* (14)*, and assume that* (18) *holds. Then, there is at least one blow-up u_0 of u at* 0 *such that the contact set $\{u_0=0\}$ has nonempty interior.*

Proof Let $r_k\to 0$ be a sequence along which

$$\lim_{r_k\to 0} \frac{\big|\{u=0\}\cap B_{r_k}\big|}{|B_{r_k}|} \geq \theta > 0.$$

Such sequence exists (with $\theta>0$ small enough) by assumption (18).

Recall that, thanks to Proposition 6, there exists a subsequence $r_{k_j}\downarrow 0$ along which $u_{r_{k_j}}\to u_0$ uniformly on compact sets of $\mathbb{R}^n$, where $u_r(x)=r^{-2}u(rx)$ and u_0 is convex.

Assume by contradiction that $\{u_0=0\}$ has empty interior. Then, by convexity, we have that $\{u_0=0\}$ is contained in a hyperplane, say $\{u_0=0\}\subset\{x_1=0\}$.

Since $u_0>0$ in $\{x_1\neq 0\}$ and u_0 is continuous, we have that for each $\delta>0$

$$u_0 \geq \varepsilon > 0 \quad \text{in } \{|x_1|>\delta\}\cap B_1$$

for some $\varepsilon>0$.

Therefore, by uniform convergence of $u_{r_{k_j}}$ to u_0 in B_1, there is $r_{k_j}>0$ small enough such that

$$u_{r_{k_j}} \geq \frac{\varepsilon}{2} > 0 \quad \text{in } \{|x_1|>\delta\}\cap B_1.$$

In particular, the contact set of $u_{r_{k_j}}$ is contained in $\{|x_1|\leq\delta\}\cap B_1$, so

$$\frac{\left|\{u_{r_{k_j}}=0\}\cap B_1\right|}{|B_1|}\le\frac{\left|\{|x_1|\le\delta\}\cap B_1\right|}{|B_1|}\le C\delta.$$

Rescaling back to u, we find

$$\frac{\left|\{u=0\}\cap B_{r_{k_j}}\right|}{|B_{r_{k_j}}|}=\frac{\left|\{u_{r_{k_j}}=0\}\cap B_1\right|}{|B_1|}<C\delta.$$

Since we can do this for every $\delta>0$, we find that $\lim_{r_{k_j}\to 0}\frac{|\{u=0\}\cap B_{r_{k_j}}|}{|B_{r_{k_j}}|}=0$, a contradiction. Thus, the lemma is proved. □

Combining the previous lemma with the classification of blow-ups from the previous Section, we deduce:

Corollary 2 *Let u be any solution to* (14)*, and assume that* (18) *holds. Then, there is at least one blow-up of u at* 0 *of the form*

$$u_0(x)=\frac{1}{2}(x\cdot e)_+^2,\qquad e\in\mathbb{S}^{n-1}. \tag{19}$$

Proof The result follows from Lemma 5 and Theorem 4. □

We now want to use this information to show that the free boundary must be smooth in a neighborhood of 0. For this, we start with the following.

Proposition 7 *Let u be any solution to* (14)*, and assume that* (18) *holds. Fix any $\varepsilon>0$. Then, there exist $e\in\mathbb{S}^{n-1}$ and $r_\circ>0$ such that*

$$\left|u_{r_\circ}(x)-\tfrac{1}{2}(x\cdot e)_+^2\right|\le\varepsilon\qquad in\quad B_1,$$

and

$$\left|\partial_\tau u_{r_\circ}(x)-(x\cdot e)_+(\tau\cdot e)\right|\le\varepsilon\qquad in\quad B_1$$

for all $\tau\in\mathbb{S}^{n-1}$.

Proof By Corollary 2 and Proposition 6, we know that there is a subsequence $r_j\to 0$ for which $u_{r_j}\to\frac{1}{2}(x\cdot e)_+^2$ in $C^1_{\rm loc}(\mathbb{R}^n)$, for some $e\in\mathbb{S}^{n-1}$. In particular, for every $\tau\in\mathbb{S}^{n-1}$ we have $u_{r_j}\to\frac{1}{2}(x\cdot e)_+^2$ and $\partial_\tau u_{r_j}\to\partial_\tau\left[\frac{1}{2}(x\cdot e)_+^2\right]$ uniformly in B_1.

This means that, given $\varepsilon>0$, there exists $j_\circ$ such that

$$\left|u_{r_{j_\circ}}(x)-\tfrac{1}{2}(x\cdot e)_+^2\right|\le\varepsilon\qquad \text{in}\quad B_1,$$

and

$$\left|\partial_\tau u_{r_{j_\circ}}(x)-\partial_\tau\left[\tfrac{1}{2}(x\cdot e)_+^2\right]\right|\le\varepsilon\qquad \text{in}\quad B_1.$$

Since $\partial_\tau\left[\frac{1}{2}(x\cdot e)_+^2\right]=(x\cdot e)_+(\tau\cdot e)$, then the proposition is proved. □

Now, notice that if $(\tau\cdot e)>0$, then the derivatives $\partial_\tau u_0=(x\cdot e)_+(\tau\cdot e)$ are *nonnegative*, and strictly positive in $\{x\cdot e>0\}$ (see Figure 8).

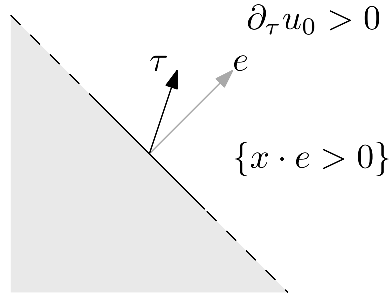

Fig. 8 Derivatives $\partial_\tau u_0$ are nonnegative if $\tau \cdot e \geq \frac{1}{2}$.

We want to transfer this information to $u_{r_\circ}$, and prove that $\partial_\tau u_{r_\circ} \geq 0$ in B_1 for all $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \geq \frac{1}{2}$. For this, we need a lemma.

Lemma 6 *Let u be any solution to* (14)*, and consider $u_{r_\circ}(x) = r_\circ^{-2} u(r_\circ x)$ and $\Omega = \{u_{r_\circ} > 0\}$.*

Assume that a function $w \in C(B_1)$ satisfies:

(a) w is bounded and harmonic in $\Omega \cap B_1$.
(b) $w = 0$ on $\partial\Omega \cap B_1$.
(c) Denoting $N_\delta := \{x \in B_1 : \mathrm{dist}(x, \partial\Omega) < \delta\}$, we have

$$w \geq -c_1 \quad \text{in} \quad N_\delta \qquad \text{and} \qquad w \geq C_2 > 0 \quad \text{in} \quad \Omega \setminus N_\delta.$$

Then, if c_1/C_2 is small enough, and $\delta > 0$ is small enough, we deduce that $w \geq 0$ in $B_{1/2} \cap \Omega$.

Proof Notice that in $\Omega \setminus N_\delta$ we already know that $w > 0$. Let $y_\circ \in N_\delta \cap \Omega \cap B_{1/2}$, and assume by contradiction that $w(y_0) < 0$.

Consider, in $B_{1/4}(y_\circ)$, the function

$$v(x) = w(x) - \gamma \left\{ u_{r_\circ}(x) - \frac{1}{2n}|x - y_\circ|^2 \right\}.$$

Then, $\Delta v = 0$ in $B_{1/4}(y_\circ) \cap \Omega$, and $v(y_\circ) < 0$. Thus, v must have a negative minimum in $\partial\big(B_{1/4}(y_\circ) \cap \Omega\big)$.

However, if c_1/C_2 and δ are small enough, then we reach a contradiction as follows. On $\partial\Omega$ we have $v \geq 0$. On $\partial B_{1/4}(y_\circ) \cap N_\delta$ we have

$$v \geq -c_1 - C_\circ \gamma \delta^2 + \frac{\gamma}{2n}\left(\frac{1}{4}\right)^2 \geq 0 \quad \text{on} \quad \partial B_{1/4}(y_\circ) \cap N_\delta.$$

On $\partial B_{1/4}(y_\circ) \cap \big(\Omega \setminus N_\delta\big)$ we have

$$v \geq C_2 - C_\circ \gamma \geq 0 \quad \text{on} \quad \partial B_{1/4}(y_\circ) \cap \big(\Omega \setminus N_\delta\big).$$

Here, we used that $\|u_{r_\circ}\|_{C^{1,1}(B_1)} \le C_\circ$, and chose $C_\circ c_1 \le \gamma \le C_2/C_\circ$. □

Using the previous lemma, we can now show that there is a cone of directions τ in which the solution is monotone near the origin.

Proposition 8 *Let u be any solution to* (14)*, and assume that* (18) *holds. Let $u_r(x) = r^{-2}u(rx)$. Then, there exist $r_\circ > 0$ and $e \in \mathbb{S}^{n-1}$ such that*

$$\partial_\tau u_{r_\circ} \ge 0 \quad in \quad B_{1/2}$$

for every $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \ge \frac{1}{2}$.

Proof By Proposition 7, for any $\varepsilon > 0$ there exist $e \in \mathbb{S}^{n-1}$ and $r_\circ > 0$ such that

$$\left|u_{r_\circ}(x) - \tfrac{1}{2}(x \cdot e)_+^2\right| \le \varepsilon \qquad \text{in} \quad B_1 \tag{20}$$

and

$$\left|\partial_\tau u_{r_\circ}(x) - (x \cdot e)_+(\tau \cdot e)\right| \le \varepsilon \qquad \text{in} \quad B_1 \tag{21}$$

for all $\tau \in \mathbb{S}^{n-1}$.

We now want to use Lemma 6 to deduce that $\partial_\tau u_{r_\circ} \ge 0$ if $\tau \cdot e \ge \frac{1}{2}$.

First, we claim that

$$u_{r_\circ} > 0 \quad \text{in} \quad \{x \cdot e > C_\circ\sqrt{\varepsilon}\},$$

$$u_{r_\circ} = 0 \quad \text{in} \quad \{x \cdot e < -C_\circ\sqrt{\varepsilon}\}, \tag{22}$$

and therefore the free boundary $\partial\Omega = \partial\{u_{r_\circ} > 0\}$ is contained in the strip $\{|x \cdot e| \le C_\circ\sqrt{\varepsilon}\}$, for some $C_\circ$ depending only on n. To prove this, notice that if $x \cdot e > C_\circ\sqrt{\varepsilon}$ then

$$u_{r_\circ} > \frac{1}{2}(C_\circ\sqrt{\varepsilon})^2 - \varepsilon > 0,$$

while if there was a free boundary point $x_\circ$ in $\{x \cdot e < -C_\circ\varepsilon\}$ then by nondegeneracy we would get

$$\sup_{B_{C_\circ\sqrt{\varepsilon}}(x_\circ)} u_{r_\circ} \ge c(C_\circ\sqrt{\varepsilon})^2 > 2\varepsilon,$$

a contradiction with (20).

Therefore, we have

$$\partial\Omega \subset \{|x \cdot e| \le C_\circ\sqrt{\varepsilon}\}. \tag{23}$$

Now, for each $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \ge \frac{1}{2}$ we define

$$w := \partial_\tau u_{r_\circ}.$$

In order to use Lemma 6, we notice that

(a) w is bounded and harmonic in $\Omega \cap B_1$.
(b) $w = 0$ on $\partial\Omega \cap B_1$.

(c) Thanks to (21), if $\delta \gg \sqrt{\varepsilon}$ then w satisfies

$$w \geq -\varepsilon \quad \text{in} \quad N_\delta$$

and

$$w \geq \delta/4 > 0 \quad \text{in} \quad (\Omega \setminus N_\delta) \cap B_1.$$

(We recall $N_\delta := \{x \in B_1 : \text{dist}(x, \partial\Omega) < \delta\}$.)

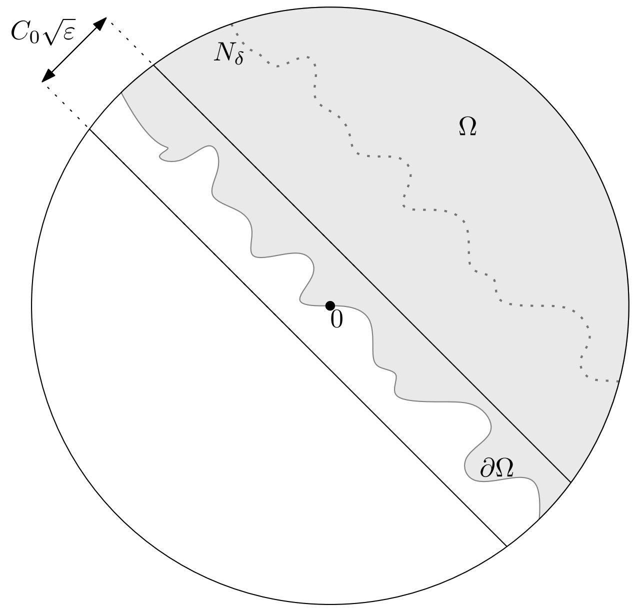

Fig. 9 The setting into which we use Lemma 6.

Indeed, to check the last inequality we use that, by (22), we have $\{x \cdot e < \delta - C_\circ\sqrt{\varepsilon}\} \subset N_\delta$. Thus, by (21), we get that for all $x \in (\Omega \setminus N_\delta) \cap B_1$

$$w \geq \frac{1}{2}(x \cdot e)_+ - \varepsilon \geq \frac{1}{2}\delta - \frac{1}{2}C_\circ\sqrt{\varepsilon} - \varepsilon \geq \frac{1}{4}\delta,$$

provided that $\delta \gg \sqrt{\varepsilon}$.

Using (a)-(b)-(c), we deduce from Lemma 6 that

$$w \geq 0 \quad \text{in} \quad B_{1/2}.$$

Since we can do this for every $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq \frac{1}{2}$, the proposition is proved. □

As a consequence of the previous proposition, we find:

Corollary 3 *Let u be any solution to* (14)*, and assume that* (18) *holds. Then, there exists $r_\circ > 0$ such that the free boundary $\partial\{u_{r_\circ} > 0\}$ is* Lipschitz *in $B_{1/2}$. In particular, the free boundary of u, $\partial\{u > 0\}$, is Lipschitz in $B_{r_\circ/2}$.*

Proof This follows from the fact that $\partial_\tau u_{r_\circ} \geq 0$ in $B_{1/2}$ for all $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e \geq \frac{1}{2}$, as explained next.

Let $x_\circ \in B_{1/2} \cap \partial\{u_{r_\circ} > 0\}$ be any free boundary point in $B_{1/2}$, and let

$$\Theta := \left\{\tau \in \mathbb{S}^{n-1} : \tau \cdot e > \tfrac{1}{2}\right\},$$

$$\Sigma_1 := \left\{x \in B_{1/2} : x = x_\circ - t\tau, \text{ with } \tau \in \Theta,\ t > 0\right\},$$

and

$$\Sigma_2 := \left\{x \in B_{1/2} : x = x_\circ + t\tau, \text{ with } \tau \in \Theta,\ t > 0\right\},$$

see Figure 10.

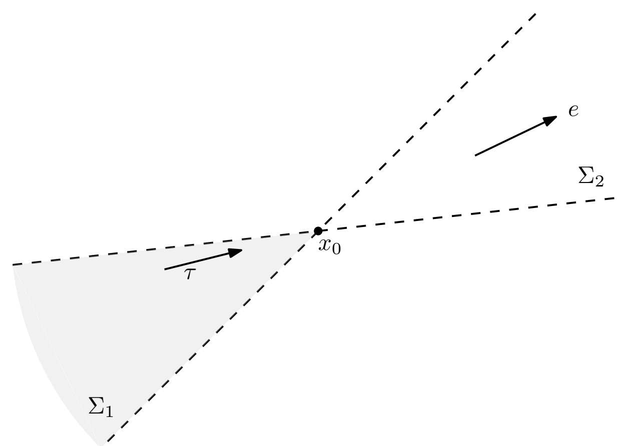

Fig. 10 Representation of Σ_1 and Σ_2.

We claim that

$$\begin{aligned} u_{r_\circ} &= 0 \quad \text{in} \quad \Sigma_1, \\ u_{r_\circ} &> 0 \quad \text{in} \quad \Sigma_2. \end{aligned} \tag{24}$$

Indeed, since $u(x_\circ) = 0$, it follows from the monotonicity property $\partial_\tau u_{r_\circ} \geq 0$ — and the nonnegativity of $u_{r_\circ}$ — that $u_{r_\circ}(x_\circ - t\tau) = 0$ for all $t > 0$ and $\tau \in \Theta$. In particular, there cannot be any free boundary point in Σ_1.

On the other hand, by the same argument, if $u_{r_\circ}(x_1) = 0$ for some $x_1 \in \Sigma_2$ then we would have $u_{r_\circ} = 0$ in $\left\{x \in B_{1/2} : x = x_1 - t\tau, \text{ with } \tau \in \Theta,\ t > 0\right\} \ni x_\circ$, and in particular $x_\circ$ would not be a free boundary point. Thus, $u_{r_\circ}(x_1) > 0$ for all $x_1 \in \Sigma_2$, and (24) is proved.

Finally, notice that (24) yields that the free boundary $\partial\{u_{r_\circ} > 0\} \cap B_{1/2}$ satisfies both the interior and exterior cone condition, and thus it is Lipschitz. □

Once we know that the free boundary is Lipschitz, we may assume without loss of generality that $e = e_n$ and that

$$\partial\{u_{r_\circ} > 0\} \cap B_{1/2} = \{x_n = g(x')\} \cap B_{1/2}$$

for a Lipschitz function $g : \mathbb{R}^{n-1} \to \mathbb{R}$. Here, $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.

Now, we want to prove that Lipschitz free boundaries are $C^{1,\alpha}$. A key ingredient for this will be the following basic property of harmonic functions (see Figure 11 for a representation of the setting).

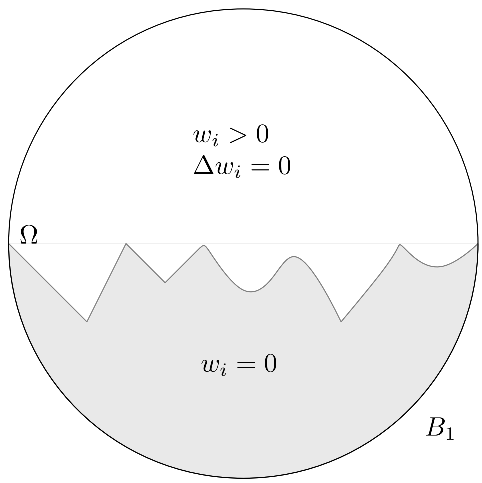

Fig. 11 Setting of the boundary Harnack.

Theorem 5 (Boundary Harnack)

Let w_1 and w_2 be positive harmonic *functions in $B_1 \cap \Omega$, where $\Omega \subset \mathbb{R}^n$ is any* Lipschitz domain.

Assume that w_1 and w_2 vanish on $\partial\Omega \cap B_1$, and $C_\circ^{-1} \le \|w_i\|_{L^\infty(B_{1/2})} \le C_\circ$ for $i = 1, 2$. Then,

$$\frac{1}{C} w_2 \le w_1 \le C w_2 \qquad \text{in} \quad \overline{\Omega} \cap B_{1/2}.$$

Moreover,

$$\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/2})} \le C$$

for some small $\alpha > 0$. The constants α and C depend only on n, $C_\circ$, and Ω.

Furthermore, if $\partial\Omega \cap B_1$ can be written as a Lipschitz graph, then C and α depend only on n, $C_\circ$, and the Lipschitz constant of Ω.

This is actually a rather difficult theorem, which does *not* follow from any explicit representation nor from Schauder-type estimates. We will not prove such theorem here; we refer to [8] for a proof of the result, as well as to [1] for a more general version of the result that allows equations with a right hand side.

Remark 3 The main point in Theorem 5 is that Ω is allowed to be *Lipschitz*. If Ω is smooth (say, C^2 or even $C^{1,\alpha}$) then it follows from a simple barrier argument that both w_1 and w_2 would be comparable to the distance to $\partial\Omega$, i.e., they vanish at a linear rate from $\partial\Omega$. However, in Lipschitz domains the result cannot be proved with a simple barrier argument, and it is much more difficult to establish.

The boundary Harnack is a crucial tool in the study of free boundary problems, and in particular in the obstacle problem. Here, we use it to prove that the free boundary is $C^{1,\alpha}$ for some small $\alpha > 0$.

Proposition 9 *Let u be any solution to* (14)*, and assume that* (18) *holds. Then, there exists* $r_\circ > 0$ *such that the free boundary* $\partial\{u_{r_\circ} > 0\}$ *is* $C^{1,\alpha}$ *in* $B_{1/4}$*, for some small* $\alpha > 0$*. In particular, the free boundary of u,* $\partial\{u > 0\}$*, is* $C^{1,\alpha}$ *in* $B_{r_\circ/4}$.

Proof Let $\Omega = \{u_{r_\circ} > 0\}$. By Corollary 3, if $r_\circ > 0$ is small enough then (possibly after a rotation) we have

$$\Omega \cap B_{1/2} = \{x_n \geq g(x')\} \cap B_{1/2}$$

and the free boundary is given by

$$\partial\Omega \cap B_{1/2} = \{x_n = g(x')\} \cap B_{1/2},$$

where g is Lipschitz.

Let

$$w_2 := \partial_{e_n} u_{r_\circ}$$

and

$$w_1 := \partial_{e_i} u_{r_\circ} + \partial_{e_n} u_{r_\circ}, \qquad i = 1, ..., n-1.$$

Since $\partial_\tau u_{r_\circ} \geq 0$ in $B_{1/2}$ for all $\tau \in \mathbb{S}^{n-1}$ with $\tau \cdot e_n \geq \frac{1}{2}$, then $w_2 \geq 0$ in $B_{1/2}$ and $w_1 \geq 0$ in $B_{1/2}$.

This is because $\partial_{e_i} + \partial_{e_n} = \partial_{e_i+e_n} = \sqrt{2}\partial_\tau$, with $\tau \cdot e_n = 1/\sqrt{2} > \frac{1}{2}$. Notice that we add the term $\partial_{e_n} u_{r_\circ}$ in w_1 in order to get a nonnegative function $w_2 \geq 0$.

Now since w_1 and w_2 are positive harmonic functions in $\Omega \cap B_{1/2}$, and vanish on $\partial\Omega \cap B_{1/2}$, we can use the boundary Harnack to get

$$\left\| \frac{w_1}{w_2} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/4})} \leq C$$

for some small $\alpha > 0$. Therefore, since $w_1/w_2 = 1 + \partial_{e_i} u_{r_\circ} / \partial_{e_n} u_{r_\circ}$, we deduce

$$\left\| \frac{\partial_{e_i} u_{r_\circ}}{\partial_{e_n} u_{r_\circ}} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/4})} \leq C. \tag{25}$$

Now, we claim that this implies that the free boundary is $C^{1,\alpha}$ in $B_{1/4}$. Indeed, if $u_{r_\circ}(x) = t$ then the normal vector to the level set $\{u_{r_\circ} = t\}$ is given by

$$\nu^i(x) = \frac{\partial_{e_i} u_{r_\circ}}{|\nabla u_{r_\circ}|} = \frac{\partial_{e_i} u_{r_\circ} / \partial_{e_n} u_{r_\circ}}{\sqrt{1 + \sum_{j=1}^n \left(\partial_{e_j} u_{r_\circ} / \partial_{e_n} u_{r_\circ}\right)^2}}, \qquad i = 1, \ldots, n.$$

This is a $C^{0,\alpha}$ function by (25), and therefore we can take $t \to 0$ to find that the free boundary is $C^{1,\alpha}$ (since the normal vector to the free boundary is given by a $C^{0,\alpha}$ function). □

So far we have proved that

$$\begin{pmatrix} \{u = 0\} \text{ has positive} \\ \text{density at the origin} \end{pmatrix} \Longrightarrow \begin{pmatrix} \text{a blow-up is} \\ u_0 = \frac{1}{2}(x \cdot e)_+^2 \end{pmatrix} \Longrightarrow \begin{pmatrix} \text{free boundary} \\ \text{is } C^{1,\alpha} \text{ near } 0 \end{pmatrix}$$

As a last step in this section, we will now prove that $C^{1,\alpha}$ free boundaries are actually C^∞.

Higher regularity of the free boundary

We want to finally prove the smoothness of free boundaries near regular points.

Theorem 6 (Smoothness of the free boundary near regular points)

Let u be any solution to (14)*, and assume that* (18) *holds. Then, the free boundary $\partial\{u > 0\}$ is C^∞ in a neighborhood of the origin.*

For this, we need the following result.

Theorem 7 (Higher order boundary Harnack)

Let $\Omega \subset \mathbb{R}^n$ be any $C^{k,\alpha}$ domain, with $k \geq 1$ and $\alpha \in (0,1)$. Let w_1, w_2 be two solutions of $\Delta w_i = 0$ in $B_1 \cap \Omega$, $w_i = 0$ on $\partial\Omega \cap B_1$, with $w_2 > 0$ in Ω.

Assume that $C_\circ^{-1} \leq \|w_i\|_{L^\infty(B_{1/2})} \leq C_\circ$. Then,

$$\left\| \frac{w_1}{w_2} \right\|_{C^{k,\alpha}(\overline{\Omega} \cap B_{1/2})} \leq C,$$

where C depends only on n, k, α, $C_\circ$, and Ω.

Contrary to Theorem 5, the proof of Theorem 7 is a perturbative argument, in the spirit of (but much more delicate than) Schauder estimates for linear elliptic equations. We will not prove the higher order boundary Harnack here; we refer to [14] for the proof of such result.

Using Theorem 7, we can finally give the:

Proof (Proof of Theorem 6) Let $u_{r_\circ}(x) = r_\circ^{-2}u(r_\circ x)$. By Proposition 9, we know that if $r_\circ > 0$ is small enough then the free boundary $\partial\{u_{r_\circ} > 0\}$ is $C^{1,\alpha}$ in B_1, and (possibly after a rotation) $\partial_{e_n} u_{r_\circ} > 0$ in $\{u_{r_\circ} > 0\} \cap B_1$. Thus, using the higher order boundary Harnack (Theorem 7) with $w_1 = \partial_{e_i} u_{r_\circ}$ and $w_2 = \partial_{e_n} u_{r_\circ}$, we find that

$$\left\| \frac{\partial_{e_i} u_{r_\circ}}{\partial_{e_n} u_{r_\circ}} \right\|_{C^{1,\alpha}(\overline{\Omega}\cap B_{1/2})} \leq C.$$

Actually, by a simple covering argument we find that

$$\left\| \frac{\partial_{e_i} u_{r_\circ}}{\partial_{e_n} u_{r_\circ}} \right\|_{C^{1,\alpha}(\overline{\Omega}\cap B_{1-\delta})} \leq C_\delta \tag{26}$$

for any $\delta > 0$.

Now, as in the proof of Proposition 9, we notice that if $u_{r_\circ}(x) = t$ then the normal vector to the level set $\{u_{r_\circ} = t\}$ is given by

$$\nu^i(x) = \frac{\partial_{e_i} u_{r_\circ}}{|\nabla u_{r_\circ}|} = \frac{\partial_{e_i} u_{r_\circ} / \partial_{e_n} u_{r_\circ}}{\sqrt{1 + \sum_{j=1}^n \left(\partial_{e_j} u_{r_\circ} / \partial_{e_n} u_{r_\circ} \right)^2}}, \qquad i = 1, \dots, n.$$

By (26), this is a $C^{1,\alpha}$ function in $B_{1-\delta}$ for any $\delta > 0$, and therefore we can take $t \to 0$ to find that the normal vector to the free boundary is $C^{1,\alpha}$ inside B_1. But this means that the free boundary is actually $C^{2,\alpha}$.

Repeating now the same argument, and using that the free boundary is $C^{2,\alpha}$ in $B_{1-\delta}$ for any $\delta > 0$, we find that

$$\left\| \frac{\partial_{e_i} u_{r_\circ}}{\partial_{e_n} u_{r_\circ}} \right\|_{C^{2,\alpha}(\overline{\Omega}\cap B_{1-\delta'})} \leq C_{\delta'},$$

which yields that the normal vector is $C^{2,\alpha}$ and thus the free boundary is $C^{3,\alpha}$. Iterating this argument, we find that the free boundary $\partial\{u_{r_\circ} > 0\}$ is C^∞ inside B_1, and hence $\partial\{u > 0\}$ is C^∞ in a neighborhood of the origin. □

This completes the study of *regular* free boundary points. It remains to understand what happens at points where the contact set has *density zero* (see e.g. Figure 5). This is the content of the next section.

7 Singular points

We finally study the behavior of the free boundary at singular points, i.e., when

$$\lim_{r \to 0} \frac{\left|\{u = 0\} \cap B_r\right|}{|B_r|} = 0. \tag{27}$$

For this, we first notice that, as a consequence of the results of the previous Section, we get the following.

Proposition 10 *Let u be any solution to* (14)*. Then, we have the following dichotomy:*

(a) Either (18) *holds and all blow-ups of u at* 0 *are of the form*

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2,$$

for some $e \in \mathbb{S}^{n-1}$.

(b) Or (27) *holds and all blow-ups of u at* 0 *are of the form*

$$u_0(x) = \frac{1}{2}x^T Ax,$$

for some matrix $A \geq 0$ *with* $\operatorname{tr} A = 1$.

Points of the type (a) were studied in the previous Section; they are called *regular* points and the free boundary is C^∞ around them (in particular, the blow-up is unique). Points of the type (b) are those at which the contact set has zero density, and are called *singular* points.

To prove the result, we need the following:

Lemma 7 *Let u be any solution to* (14)*, and assume that* (27) *holds. Then, every blow-up of u at* 0 *satisfies* $|\{u_0 = 0\}| = 0$.

Proof Let u_0 be a blow-up of u at 0, i.e., $u_{r_k} \to u_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ along a sequence $r_k \to 0$, where $u_r(x) = r^{-2}u(rx)$.

Notice that the functions u_r solve

$$\Delta u_r = \chi_{\{u_r>0\}} \quad \text{in} \quad B_1,$$

in the sense that

$$\int_{B_1} \nabla u_r \cdot \nabla \eta \, dx = \int_{B_1} \chi_{\{u_r>0\}} \eta \, dx \qquad \text{for all } \eta \in C_c^\infty(B_1). \tag{28}$$

Moreover, by assumption (27), we have $\big|\{u_r = 0\} \cap B_1\big| \longrightarrow 0$, and thus taking limits $r_k \to 0$ in (28) we deduce that $\Delta u_0 = 1$ in B_1. Since we know that u_0 is convex, nonnegative, and homogeneous, this implies that $|\{u_0 = 0\}| = 0$. □

We can now give the:

Proof (Proof of Theorem 10) By the classification of blow-ups, Theorem 4, the possible blow-ups can only have one of the two forms presented. If (18) holds for at least one blow-up, thanks to the smoothness of the free boundary (by Proposition 9), it holds for all blow-ups, and thus, by Corollary 2, $u_0(x) = \frac{1}{2}(x \cdot e)_+^2$ (and in fact, the smoothness of the free boundary yields uniqueness of the blow-up in this case).

If (27) holds, then by Lemma 7 the blow-up u_0 must satisfy $\big|\{u_0 = 0\}\big| = 0$, and thus we are in case (b) (see the proof of Theorem 4). □

In the previous Section we proved that the free boundary is C^∞ in a neighborhood of any regular point. A natural question then is to understand better the solution u near singular points. One of the main results in this direction is the following.

Theorem 8 (Uniqueness of blow-ups at singular points)

Let u be any solution to (14)*, and assume that* 0 *is a singular free boundary point.*

Then, there exists a homogeneous quadratic polynomial $p_2(x) = \frac{1}{2}x^T Ax$, with $A \geq 0$ and $\Delta p_2 = 1$, such that

$$u_r \longrightarrow p_2 \qquad in \quad C^1_{\rm loc}(\mathbb{R}^n).$$

In particular, the blow-up of u at 0 *is unique, and $u(x) = p_2(x) + o(|x|^2)$.*

To prove this, we need the following monotonicity formula due to Monneau.

Theorem 9 (Monneau's monotonicity formula)

Let u be any solution to (14)*, and assume that* 0 *is a singular free boundary point.*

Let q be any homogeneous quadratic polynomial with $q \geq 0$, $q(0) = 0$, and $\Delta q = 1$. Then, the quantity

$$M_{u,q}(r) := \frac{1}{r^{n+3}} \int_{\partial B_r} (u-q)^2$$

is monotone in r, that is, $\frac{d}{dr} M_{u,q}(r) \geq 0$.

Proof We sketch the argument here, and refer to [32, Theorem 7.4] for more details.

We first notice that

$$M_{u,q}(r) = \int_{\partial B_1} \frac{(u-q)^2(rx)}{r^4},$$

and hence a direct computation yields

$$\frac{d}{dr} M_{u,q}(r) = \frac{2}{r^{n+4}} \int_{\partial B_r} (u-q) \left\{ x \cdot \nabla(u-q) - 2(u-q) \right\}.$$

On the other hand, it turns out that

$$\frac{1}{r^{n+3}} \int_{\partial B_r} (u-q) \left\{ x \cdot \nabla(u-q) - 2(u-q) \right\} = W_u(r) - W_u(0^+) + \\ + \frac{1}{r^{n+2}} \int_{B_r} (u-q) \Delta(u-q),$$

where $W_u(r)$ (as defined in (15)) is monotone increasing in $r > 0$ thanks to Theorem 3. Thus, we have

$$\frac{d}{dr} M_{u,q}(r) \geq \frac{2}{r^{n+3}} \int_{B_r} (u-q) \Delta(u-q).$$

But since $\Delta u = \Delta q = 1$ in $\{u > 0\}$, and $(u - q)\Delta(u - q) = q \geq 0$ in $\{u = 0\}$, then we have

$$\frac{d}{dr} M_{u,q}(r) \geq \frac{2}{r^{n+3}} \int_{B_r \cap \{u=0\}} q \geq 0,$$

as wanted. □

We can now give the:

Proof (Proof of Theorem 8) By Proposition 10 (and Proposition 6), we know that at any singular point we have a subsequence $r_j \to 0$ along which $u_{r_j} \to p$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, where p is a 2-homogeneous quadratic polynomial satisfying $p(0) = 0$, $p \geq 0$, and $\Delta p = 1$. Thus, we can use Monneau's monotonicity formula with such polynomial p to find that

$$M_{u,p}(r) := \frac{1}{r^{n+3}} \int_{\partial B_r} (u - p)^2$$

is monotone increasing in $r > 0$. In particular, the limit $\lim_{r\to 0} M_{u,p}(r) =: M_{u,p}(0^+)$ exists.

Now, recall that we have a sequence $r_j \to 0$ along which $u_{r_j} \to p$. In particular, $r_j^{-2} \left\{u(r_j x) - p(r_j x)\right\} \longrightarrow 0$ locally uniformly in $\mathbb{R}^n$, i.e.,

$$\frac{1}{r_j^2} \|u - p\|_{L^\infty(B_{r_j})} \longrightarrow 0$$

as $r_j \to 0$. This yields that

$$M_{u,p}(r_j) \leq \frac{1}{r_j^{n+3}} \int_{\partial B_{r_j}} \|u - p\|^2_{L^\infty(B_{r_j})} \longrightarrow 0$$

along the subsequence $r_j \to 0$, and therefore $M_{u,p}(0^+) = 0$.

Let us show that this implies the uniqueness of blow-up. Indeed, if there was another subsequence $r_\ell \to 0$ along which $u_{r_\ell} \to q$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, for a 2-homogeneous quadratic polynomial q, then we would repeat the argument above to find that $M_{u,q}(0^+) = 0$. But then this yields, by homogeneity of p and q,

$$\int_{\partial B_1} (p - q)^2 = \frac{1}{r^{n+3}} \int_{\partial B_r} (p - q)^2 \leq 2M_{u,p}(r) + 2M_{u,q}(r) \longrightarrow 0,$$

and hence

$$\int_{\partial B_1} (p - q)^2 = 0.$$

This means that $p = q$, and thus the blow-up of u at 0 is unique.

Let us finally show that $u(x) = p(x) + o(|x|^2)$, i.e., $r^{-2}\|u - p\|_{L^\infty(B_r)} \to 0$ as $r \to 0$. Indeed, assume by contradiction that there is a subsequence $r_k \to 0$ along which

$$r_k^{-2} \|u - p\|_{L^\infty(B_{r_k})} \geq c_1 > 0.$$

Then, there would be a subsequence of r_{k_i} along which $u_{r_{k_i}} \to u_0$ in $C^1_{\rm loc}(\mathbb{R}^n)$, for a certain blow-up u_0 satisfying $\|u_0 - p\|_{L^\infty(B_1)} \geq c_1 > 0$. However, by uniqueness of blow-up it must be $u_0 = p$, and hence we reach a contradiction. □

Summarizing, we have proved the following result:

Theorem 10 *Let u be any solution to* (14). *Then, we have the following dichotomy:*

(a) Either all blow-ups of u at 0 *are of the form*

$$u_0(x) = \frac{1}{2}(x \cdot e)_+^2 \qquad \textit{for some} \quad e \in \mathbb{S}^{n-1},$$

and the free boundary is C^∞ in a neighborhood of the origin.

(b) Or there is a homogeneous quadratic polynomial p, with $p(0) = 0$, $p \geq 0$, and $\Delta p = 1$, such that

$$\|u - p\|_{L^\infty(B_r)} = o(r^2) \qquad \textit{as} \quad r \to 0.$$

In particular, when this happens we have

$$\lim_{r\to 0} \frac{\big|\{u = 0\} \cap B_r\big|}{|B_r|} = 0.$$

The last question that remains to be answered is: How large can the set of singular points be? This is the topic of the following section.

8 The size of the singular set

We finish these notes with a discussion of more recent results (as well as some open problems) about the set of singular points.

Recall that a free boundary point $x_\circ \in \partial\{u > 0\}$ is singular whenever

$$\lim_{r\to 0} \frac{\big|\{u = 0\} \cap B_r(x_\circ)\big|}{|B_r(x_\circ)|} = 0.$$

The main known result on the size of the singular set reads as follows.

Theorem 11 ([7])

Let u be any solution to (14). *Let $\Sigma \subset B_1$ be the set of singular points.*
Then, $\Sigma \cap B_{1/2}$ is contained in a C^1 manifold of dimension $n - 1$.

This result is sharp, in the sense that it is not difficult to construct examples in which the singular set is $(n - 1)$-dimensional; see [36].

As explained below, such result essentially follows from the uniqueness of blow-ups at singular points, established in the previous section.

Indeed, given any singular point $x_\circ$, let $p_{x_\circ}$ be the blow-up of u at $x_\circ$ (recall that $p_{x_\circ}$ is a nonnegative 2-homogeneous polynomial). Let k be the dimension of the set $\{p_{x_\circ} = 0\}$ — notice that this is a proper linear subspace of $\mathbb{R}^n$, so that $k \in \{0, \ldots, n-1\}$ — and define

$$\Sigma_k := \big\{x_\circ \in \Sigma : \dim(\{p_{x_\circ} = 0\}) = k\big\}. \tag{29}$$

Clearly, $\Sigma = \cup_{k=0}^{n-1} \Sigma_k$.

The following result gives a more precise description of the singular set.

Proposition 11 ([7])

Let u be any solution to (14)*. Let $\Sigma_k \subset B_1$ be defined by* (29)*, $k = 1, \ldots, n-1$. Then, $\Sigma_k \cap B_{1/2}$ is contained in a C^1 manifold of dimension k.*

The rough heuristic idea of the proof of this result is as follows. Assume for simplicity that $n = 2$, so that $\Sigma = \Sigma_1 \cup \Sigma_0$.

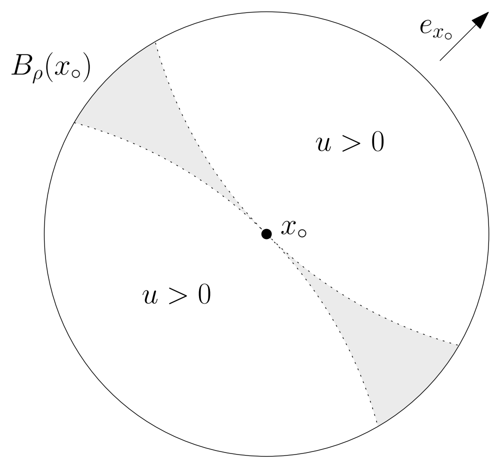

Fig. 12 u is positive in $\{x \in B_\rho(x_\circ) : |(x - x_\circ) \cdot e_{x_\circ}| > \omega(|x - x_\circ|)\}$.

Let us take a point $x_\circ \in \Sigma_0$. Then, by Theorem 10, we have the expansion

$$u(x) = p_{x_\circ}(x - x_\circ) + o\big(|x - x_\circ|^2\big) \tag{30}$$

where $p_{x_\circ}$ is the blow-up of u at $x_\circ$ (recall that this came from the uniqueness of blow-up at $x_\circ$). By definition of Σ_0, the polynomial $p_{x_\circ}$ must be positive outside the origin, and thus by homogeneity satisfies $p_{x_\circ}(x - x_\circ) \ge c|x - x_\circ|^2$, with $c > 0$. This, combined with (30), yields then that u must be positive in a neighborhood of the origin. In particular, all points in Σ_0 are isolated.

On the other hand, let us now take a point $x_\circ \in \Sigma_1$. Then, by definition of Σ_1 the blow-up must necessarily be of the form $p_{x_\circ}(x) = \frac{1}{2}(x \cdot e_{x_\circ})^2$, for some $e_{x_\circ} \in \mathbb{S}^{n-1}$. Again by the expansion (30), we find that u is positive in a region of the form

$$\left\{x \in B_\rho(x_\circ) : \left|(x - x_\circ) \cdot e_{x_\circ}\right| > \omega(|x - x_\circ|)\right\},$$

where ω is a certain modulus of continuity, and $\rho > 0$ is small (see Figure 12).

This is roughly saying that the set Σ_1 “has a tangent plane” at $x_\circ$. Repeating the same at any other point $\tilde{x}_\circ \in \Sigma_1$ we find that the same happens at every point in Σ_1 and, moreover, if $\tilde{x}_\circ$ is close to $x_\circ$ then $e_{\tilde{x}_\circ}$ must be close to $e_{x_\circ}$ — otherwise the expansions (30) at $\tilde{x}_\circ$ and $x_\circ$ would not match. Finally, since the modulus ω can be made independent of the point (by a compactness argument), then it turns out that the set Σ_1 is contained in a C^1 curve (see Figure 13).

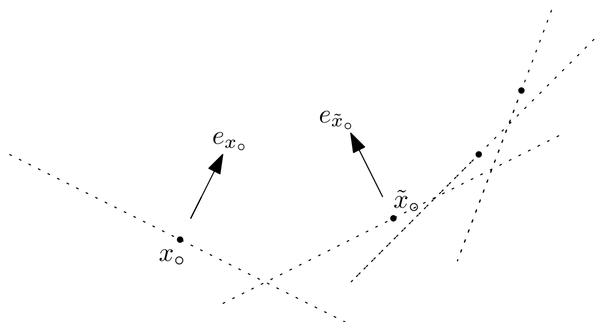

Fig. 13 Singular points $x_\circ, \tilde{x}_\circ \in \Sigma_1$.

What we discussed here is just an heuristic argument; the actual proof uses Whitney’s extension theorem and can be found for example in [32].

We finally discuss some recent results [19] on the fine structure and regularity of singular points.

9 Finer understanding of singular points

In order to get a finer understanding of singular points, we will need to relate the obstacle problem to the so-called *thin obstacle problem*. For this, we first briefly introduce such free boundary problem and summarize the main known results in this context.

The thin obstacle problem

The thin obstacle problem is another classical free boundary problem, which was originally studied by Signorini in connection with linear elasticity [38, 39]. The problem gained further attention in the seventies due to its connection to mechanics, biology, and even finance — see [16], [28, 11], and [34] —, and since then it has been widely studied in the mathematical community; see [6, 2, 3, 22, 32, 26, 13, 4, 20, 10, 17, 37] and references therein.

We say that $w \in H^1(B_1)$ is a solution to the thin obstacle problem (with zero obstacle) if

$$\begin{cases} -\Delta w = 0 & \text{in } B_1 \setminus (\{x_n = 0\} \cap \{w = 0\}) \\ -\Delta w \geq 0 & \text{in } B_1 \\ \quad w \geq 0 & \text{on } \{x_n = 0\} \\ \quad w = g & \text{on } \partial B_1, \end{cases} \tag{31}$$

in the weak sense, for some boundary data $g \in C^0(\partial B_1 \cap \{x_n \geq 0\})$. These solutions are minimizers of the Dirichlet energy

$$\int_{B_1} |\nabla w|^2,$$

under the constrain $w \geq 0$ on $\{x_n = 0\}$, and with boundary conditions $w = g$ on ∂B_1.

The contact set is denoted by

$$\Lambda(w) := \left\{x' \in \mathbb{R}^{n-1} : w(x', 0) = 0\right\},$$

and the free boundary is $\Gamma(w) = \partial\Lambda(w)$. Here, we denoted $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Solutions to (31) are $C^{1,\frac{1}{2}}$ (see [2]), and this is optimal.

A key tool in establishing the optimal regularity of solutions is the Almgren frequency formula:

$$N_w(r) := \frac{r\int_{B_r} |\nabla w|^2}{\int_{\partial B_r} w^2} \qquad \text{is monotone in } r, \tag{32}$$

namely, $N_w'(r) \geq 0$ for $r \in (0, 1)$.

This allows one to show that *blow-ups are homogeneous*, and the degree of homogeneity of any blow-up of w at 0 is exactly $N_w(0^+)$. Moreover, it was shown in [3] that the lowest possible homogeneity is in fact 3/2, and this gives the optimal $C^{1,\frac{1}{2}}$ regularity of solutions; we refer to [3] for more details.

The free boundary in the thin obstacle problem

The main known results concerning the structure of the free boundary are the following.

The free boundary can be divided into two sets,

$$\Gamma(w) = \mathrm{Reg}(w) \cup \mathrm{Deg}(w),$$

the set of *regular points*,

$$\mathrm{Reg}(w) := \left\{ x \in \Gamma(w) : 0 < cr^{3/2} \le \sup_{B_r(x)} w \le Cr^{3/2}, \quad \forall r \in (0, r_\circ) \right\},$$

and the set of non-regular points or *degenerate points*

$$\mathrm{Deg}(w) := \left\{ x \in \Gamma(w) : 0 \le \sup_{B_r(x)} w \le Cr^2, \quad \forall r \in (0, r_\circ) \right\}, \tag{33}$$

Alternatively, each of the subsets can be defined according to the homogeneity of the blow-up at that point. Indeed, the set of regular points are those whose blow-up is of order $\frac{3}{2}$, and the set of degenerate points are those whose blow-up is of order κ for some $\kappa \in [2, \infty)$.

Let us denote Γ_κ the set of free boundary points of order κ. That is, those points whose blow-up is homogeneous of order κ. Then, the free boundary can be divided as

$$\Gamma(w) = \Gamma_{3/2} \cup \Gamma_{\mathrm{even}} \cup \Gamma_{\mathrm{odd}} \cup \Gamma_{\mathrm{half}} \cup \Gamma_*, \tag{34}$$

where:

- $\Gamma_{3/2} = \mathrm{Reg}(w)$ is the set of regular points. They are an open $(n-2)$-dimensional subset of $\Gamma(w)$, and it is C^∞ (see [3] and [26, 15]).
- $\Gamma_{\mathrm{even}} = \bigcup_{m\ge 1} \Gamma_{2m}(w)$ denotes the set of points whose blow-ups have even homogeneity. Equivalently, they can also be characterised as those points of the free boundary where the contact set has zero density, and they are often called singular points. They are contained in the countable union of C^1 $(n-2)$-dimensional manifolds; see [22].
- $\Gamma_{\mathrm{odd}} = \bigcup_{m\ge 1} \Gamma_{2m+1}(w)$ is also an at most $(n-2)$-dimensional subset of the free boundary and it is $(n-2)$-rectifiable (see [20]). They can be characterised as those points of the free boundary where the contact set has density one.
- $\Gamma_{\mathrm{half}} = \bigcup_{m\ge 1} \Gamma_{2m+3/2}(w)$ corresponds to those points with blow-up of order $\frac{7}{2}$, $\frac{11}{2}$, etc. They are much less understood than regular points. The set Γ_{half} is an $(n-2)$-dimensional subset of the free boundary and it is a $(n-2)$-rectifiable set (see [20]).
- Γ_* is the set of all points with homogeneities $\kappa \in (2, \infty)$, with $\kappa \notin \mathbb{N}$ and $\kappa \notin 2\mathbb{N} - \frac{1}{2}$. This set has Hausdorff dimension at most $n-3$, so it is always *small*.

Remark 4 It is interesting to notice that, if w solves the Signorini problem (31), then $U(x) = \{w, -w\}$ is a $C^{1,\mu}$ two-valued harmonic function, to which the results of [27] apply. In particular, most of the above results for (31) follow from [27].

Dimension-reduction in the thin obstacle problem

Dimension-reduction arguments were first introduced by Almgren in the context of minimal surfaces, and are nowadays used in a variety of settings in PDEs. We will later describe some recent results [19] on the obstacle problem that are based on such kind of arguments.

Now, in order to present these ideas in a simpler context, we will show the following result for the thin obstacle problem:

Theorem 12 *Let w be any solution to the thin obstacle problem. Then,*

$$\dim_{\mathcal{H}}(\Gamma_*) \le n-3.$$

In order to prove Theorem 12, we follow the arguments of [41]. A key tool is the Almgren frequency function

$$N_{w,x}(r) := \frac{r\int_{B_r(x)} |\nabla w|^2}{\int_{\partial B_r(x)} w^2}, \tag{35}$$

which is monotone in r. In particular, we can define

$$N_{w,x}(0+) := \lim_{r\to 0} N_{w,x}(r).$$

It is not difficult to show that such quantity is upper semicontinuous in x, that is, if $w_j \to w$ uniformly in B_1 and $x_j \to x$ then

$$\limsup_j N_{w_j,x_j}(0+) \le N_{w,x}(0+).$$

Notice also that, if q is any global μ-homogeneous solution of the thin obstacle problem, then:

- $N_{q,0}(r) \equiv \mu$ is constant in r.
- $N_{q,x}(0+) \le \mu$ for all $x \in \{x_n = 0\}$.
- The set $S(q) = \{x : N_{q,x}(0+) = \mu\}$ is a linear subspace, and $q(x+y) = q(y)$ for all $y \in \mathbb{R}^n$ and $x \in S(q)$.
- If $\mu \notin \mathbb{N}$ and $\mu \notin 2\mathbb{N} - \frac{1}{2}$, then $S(q)$ is of dimension at most $n-3$.

The last property follows simply from the fact that the only possible homogeneities μ for the thin obstacle problem in dimension $n = 2$ are $\mu = \frac{3}{2}, \frac{7}{2}, \frac{11}{2}, \ldots$ or $\mu = 2, 3, 4, 5, \ldots$; see for example [22].

Using this, we will now prove:

Lemma 8 *Let w be any solution to the thin obstacle problem and $x \in \Gamma_*$.*

Then, for every $\delta > 0$ there exists $\varepsilon > 0$ and $\rho > 0$ (depending on u, x, δ) such that for every $r \in (0, \rho]$ there exists an $(n-3)$-dimensional plane $L_{x,r} \subset \{x_n = 0\}$, passing through x, such that

$$B_r(x) \cap \{y : N_{w,y}(0+) \ge N_{w,x}(0+) - \varepsilon\} \subset \{y : \mathrm{dist}(y, L_{x,r}) < \delta r\}.$$

Proof We prove the result for $x = 0$. Denote

$$w_r(x) := \frac{w(rx)}{\left(\fint_{\partial B_r} w^2\right)^{1/2}}.$$

Recall that w_r converges along subsequences to homogeneous blow-ups q, which are homogeneous of degree $\mu = N_{w,x}(0+)$. (Notice that a priori different subsequences could lead to different blow-ups, but they all have the same homogeneity.)

Assume by contradiction that for some $\delta > 0$, $\varepsilon_k \downarrow 0$, and $r_k \downarrow 0$, we have

$$B_{r_k} \cap \{y : N_{w,y}(0+) \geq \mu - \varepsilon_k\} \not\subset \{y : \operatorname{dist}(y, L) < \delta r_k\}$$

for every $(n-3)$-dimensional linear subspace L of $\{x_n = 0\}$. By scaling,

$$B_1 \cap \{y : N_{w_{r_k},y}(0+) \geq \mu - \varepsilon_k\} \not\subset \{y : \operatorname{dist}(y, L) < \delta\} \tag{36}$$

for every $(n-2)$-dimensional linear subspace L of $\{x_n = 0\}$. By $C^{1,\alpha}$ estimates and the Arzela-Ascoli theorem, after passing to a subsequence, w_{r_k} converges to a blow-up q in the C^1 norm on each compact subset of $\{x_n \geq 0\}$. Since $0 \in \Gamma_*$, then $\dim S(q) \leq n-2$, so we can take $L = S(q)$ (or any $(n-3)$-dimensional linear subspace containing $S(q)$.

By (36), there exists z_k with $N_{w_{r_k},z_k}(0+) \geq \mu - \varepsilon_k$ and $\operatorname{dist}(z_k, L) \geq \delta$. After passing to a subsequence, $z_k \to z$. By the upper semicontinuity of frequency, $N_{q,z}(0+) \geq \mu$, implying that $z \in S(q)$. But $\operatorname{dist}(z, S(q)) \geq \operatorname{dist}(z, L) \geq \delta$, a contradiction. □

We will also need the following result, whose proof can be found for example in [41].

Proposition 12 *Let $E \subseteq \mathbb{R}^n$ such that for each $\delta > 0$, each $x \in E$, and each $r > 0$ small enough, there exists an m-dimensional plane $L_{x,r}$, passing through x, for which*

$$E \cap B_r(x) \subset \{y : \operatorname{dist}(y, L_{x,r}) < \delta r\}.$$

Then, $\dim_{\mathcal{H}}(E) \leq m$.

For convenience of the reader, we provide a proof of Proposition 12 in the Appendix.

Using this, we can give the:

Proof (Proof of Theorem 12) Let $\delta > 0$. For $i = 1, 2, 3, \ldots$, define $\Gamma_*^{(i)}$ to be the set of all points $y \in \Gamma_*$ such that the conclusion of Lemma 8 holds true with $\varepsilon = 1/i$ and $\rho = 1/i$. Observe that $\Gamma_* = \bigcup_i \Gamma_*^{(i)}$. For each $j = 1, 2, 3, \ldots$, define

$$\Gamma_*^{(i,j)} = \{x \in \Gamma_*^{(i)} : (j-1)/i < N_{w,x}(0+) \leq j/i\}.$$

Observe that $\Gamma_* = \bigcup_{i,j} \Gamma_*^{(i,j)}$. Moreover, for every $x \in \Gamma_*^{(i,j)}$,

$$\Gamma_*^{(i,j)} \subset \{y : N_{w,y}(0+) > N_{w,x}(0+) - 1/i\}$$

and thus by Lemma 8 for every $r \in (0, 1/i]$ there exists a $(n-3)$-dimensional plane $L_{x,r}$ of $\{x_n = 0\}$, passing through x, such that

$$\Gamma_*^{(i,j)} \cap B_r(x) \subset \{y : \mathrm{dist}(y, L_{x,r}) < \delta r\}.$$

Since $\delta > 0$ is arbitrary, by Proposition 12 with $E = \Gamma_*^{(i,j)}$, we have $\dim_{\mathcal{H}}(\Gamma_*^{(i,j)}) \le m$, and thus Γ_* has Hausdorff dimension at most m. □

Relating the obstacle problem and the thin obstacle problem

A key idea in [19] is to notice that, if u is a solution to the obstacle problem, 0 is a singular point, and we consider

$$w = u - p_2,$$

where p_2 is the blow-up of u at 0, then w behaves like a solution to the thin obstacle problem.

Indeed, since $\Delta p_2 = 1$ then $\Delta w = -\chi_{\{u=0\}}$, and therefore w solves

$$\begin{cases} -\Delta w = 0 & \text{in } B_1 \setminus \{u = 0\} \\ -\Delta w \ge 0 & \text{in } B_1 \\ \quad w \ge 0 & \text{on } \{p_2 = 0\}. \end{cases}$$

When p_2 is of the form $p_2(x) = \frac{1}{2}(x_n)^2$, then as we rescale w closer and closer to the origin, it turns out that $\{u = 0\}$ becomes closer and closer to $\{p_2 = 0\} = \{x_n = 0\}$, and thus w becomes closer and closer to a solution to the Signorini problem (or simply an harmonic function if $\{p_2 = 0\}$ is too small).

To make this argument precise, we need the following.

Proposition 13 *Let u be a solution to the obstacle problem in B_1, and assume that 0 is a singular point. Let p_2 be the blow-up of u at 0, and let*

$$w := u - p_2$$

and

$$N_w(r) := \frac{r\int_{B_r} |\nabla w|^2}{\int_{\partial B_r} w^2}. \tag{37}$$

Then for all $r \in (0,1)$ we have $N_w(r) \ge 2$ and

$$N_w'(r) \ge \frac{2}{r}\frac{\left(r\int_{B_r} w\Delta w\right)^2}{\left(\int_{\partial B_r} w^2\right)^2} \ge 0, \quad \forall r \in (0,1).$$

Proof Let

$$H(r) = r^{1-n} \int_{\partial B_r} w^2, \quad D(r) = r^{2-n} \int_{\partial B_r} |\nabla w|^2.$$

Then, we have

$$H'(1) = 2 \int_{\partial B_1} w w_\nu$$

$$D'(1) = 2 \int_{\partial B_1} w_\nu^2 - \int_{B_1} (x \cdot \nabla w) \Delta w$$

$$D(r) = \int_{\partial B_1} w w_\nu - \int_{B_1} w \Delta w$$

We first claim that

$$D(r) \geq 2H(r).$$

Indeed, thanks to Weiss' monotonicity formula (Theorem 3) we have

$$\begin{aligned} 0 \leq W_u(r) - W_u(0^+) &= W_u(r) - W_{p_2}(r) \\ &= \text{(Exercise)} \\ &= r^{-2-n} \int_{\partial B_r} |\nabla w|^2 - 2r^{-3-n} \int_{\partial B_r} w^2 \\ &= \frac{1}{r^4} \big(D(r) - 2H(r) \big), \end{aligned}$$

and thus the claim follows.

On the other hand, since

$$N_w(r) = \frac{D(r)}{H(r)}$$

we then have

$$\begin{aligned} N_w'(1) &= \frac{D'(1)H(1) - H'(1)D(1)}{H(1)^2} \\ &= \frac{2\big(\int_{\partial B_1} w_\nu^2 - \int_{B_1} (x \cdot \nabla w)\Delta w \big) \int_{\partial B_1} w^2 - 2 \int_{\partial B_1} w w_\nu \big(\int_{\partial B_1} w w_\nu - \int_{B_1} w \Delta w \big)}{H(1)^2} \\ &= 2 \frac{\int_{\partial B_1} w_\nu^2 \int_{\partial B_1} w^2 - \big(\int_{\partial B_1} w w_\nu \big)^2 + \text{rest}}{H(1)}. \end{aligned}$$

where

$$\begin{aligned} \text{rest} :&= \left(\int_{B_1} w \Delta w \right) \left(\int_{\partial B_1} w w_\nu \right) - \left(\int_{B_1} (x \cdot \nabla w) \Delta w \right) \int_{\partial B_1} w^2 \\ &= \left(\int_{B_1} w \Delta w \right)^2 + \left(\int_{B_1} w \Delta w \right) D(1) - \left(\int_{B_1} (x \cdot \nabla w) \Delta w \right) H(1). \end{aligned}$$

But now recall that $\Delta w = -\chi_{\{u=0\}}$ and $(x \cdot \nabla w) = -2x \cdot \nabla p = -2p$ on $\{u = 0\}$ thus we obtain

$$\int_{B_1} w\Delta w = \int_{\{u=0\}\cap B_1} p \geq 0 \qquad \int_{B_1} (x \cdot \nabla w)\Delta w = 2\int_{\{u=0\}\cap B_1} p \geq 0$$

Using this we get

$$\text{rest} = \left(\int_{B_1} w\Delta w\right)^2 + \big(D(1) - 2H(1)\big)\int_{\{u=0\}\cap B_1} p \geq 0,$$

where we used that $D(1) \geq 2H(1)$. ◻

The previous proposition allows us to show the following. Recall that the sets Σ_k were defined in (29).

Corollary 4 *Let u be a solution to the obstacle problem in B_1, and assume that 0 is a singular point. Let p_2 be the blow-up of u at 0, and let*

$$w := u - p_2$$

and

$$w_r(x) := \frac{w(rx)}{r^{(1-n)/2}\|w\|_{L^2(\partial B_r)}}. \tag{38}$$

Then, $w_{r_j} \to q$ in $L^2_{\text{loc}}(\mathbb{R}^n)$ along a subsequence $r_j \to 0$, and:

- *If $0 \in \Sigma_{n-1}$, then we have that q is a homogeneous solution to the Signorini problem of degree > 2.*
- *If $0 \in \Sigma_k$, $k \leq n - 2$, then we have that q is a homogeneous harmonic function of degree ≥ 2.*

Moreover, we also have the following.

Proposition 14 *Let u be a solution to the obstacle problem in B_1, and assume that 0 is a singular point. Let p_2 be the blow-up of u at 0, and let*

$$w := u - p_2$$

and $N_w(r)$ be given by (37).

Assume that $N_w(0^+) \geq \kappa$. Then, the quantity

$$\frac{1}{r^{n-1+2\kappa}}\int_{\partial B_r} (u - p_2)^2$$

is monotone nondecreasing for $r \in (0, 1)$.

Proof Denote, as before,

$$H(r) = r^{1-n}\int_{\partial B_r} w^2.$$

Then,

$$\frac{d}{dr}\left\{\frac{1}{r^{2\kappa}}H(r)\right\} = \frac{1}{r^{2\kappa+1}}\big(rH'(r) - 2\kappa\, H(r)\big).$$

Now, using that

$$N_w(r) = \frac{D(r)}{H(r)} = \frac{\frac{1}{2}rH'(r) - r^{2-n}\int_{B_r} w\Delta w}{H(r)},$$

that $w\Delta w \geq 0$, and that $N_w(r)$ is monotone, we have

$$\kappa \leq N_w(0^+) \leq N_w(r) \leq \frac{\frac{1}{2}rH'(r)}{H(r)}.$$

This yields $rH'(r) \geq 2\kappa\, H(r)$, and the result follows. □

Notice that the previous result is an improved Monneau-type monotonicity formula, which gives a finer information at singular points whenever $\kappa > 2$.

Finally, it was proved in [19] with a dimension reduction argument that the following holds.

Theorem 13 *Let u be a solution to the obstacle problem in B_1. Let p_{2,x_0} be the blow-up of u at x_0, and let*

$$w_{x_0}(x) := u(x_0 + x) - p_{2,x_0}(x)$$

and

$$N_{w,x_0}(r) := \frac{r\int_{B_r} |\nabla w_{x_0}|^2}{\int_{\partial B_r} w_{x_0}^2}.$$

Then, outside a set of Hausdorff dimension $n-3$, we have $N_{w,x_0}(0^+) \geq 3$.

Notice that the analysis is quite different in Σ_{n-1} or in Σ_k for $k \leq n-2$. In the first case, one must show that the set of points $x_0 \in \Sigma_{n-1}$ at which the blow-up of w_{x_0} is a solution to the thin obstacle problem with homogeneity in the interval $(2, 3)$ is small. This is very similar to what happens in Theorem 12. In the second case, instead, one must show that the set of points $x_0 \in \Sigma_k$ at which the blow-up of w_{x_0} is homogeneous of degree 2 is small, and the corresponding argument is different.

As a consequence of Theorem 13, it was shown in [19] that the singular set is actually contained in a $C^{1,1}$ manifold, outside a (relatively open) set of Hausdorff dimension $n-3$.

Finally, an alternative way to state the above result is that, outside a set of Hausdorff dimension $n-3$, we have

$$u(x) = p_{2,x_0}(x - x_0) + O\big(|x - x_0|^3\big).$$

This gives the sharp rate of convergence of blow-ups for the obstacle problem in $\mathbb{R}^n$; we refer to [19] for more details.

Apendix: Proof of Proposition 12

Proposition 12 will follow from the following.

Lemma 9 *For every $\beta > 0$ there exists $\delta > 0$ such that the following holds.*

Let $E \subseteq \mathbb{R}^n$ such that for each $x \in E$ and $r \in (0, r_0)$ there exists a m-dimensional plane $L_{x,r}$, passing through x, for which

$$E \cap B_r(x) \subset \{y : \mathrm{dist}(y, L_{x,r}) < \delta r\}.$$

Then, $\mathcal{H}^{m+\beta}(A) = 0$.

Proof By a covering argument, we may assume that $E \subseteq B_1$ and $0 \in E$. By assumption, there exists a plane $L_{0,1}$ such that

$$E \cap B_1 \subset \{y : \mathrm{dist}(y, L_{0,1}) < \delta\}.$$

Cover $L_{0,1}$ by a finite collection of balls $\{B_{\delta/2}(z_k)\}_{k=1,2,\ldots,N}$ where $z_k \in L_{0,1}$ for each k and $N \leq C\delta^{-m}$. Observe that $\{B_{\delta/2}(z_k)\}_{k=1,2,\ldots,N}$ covers $\{y : \mathrm{dist}(y, L_{0,1}) < \delta\}$ and thus covers $E \cap B_1$. Throw away the balls $B_{\delta/2}(z_k)$ that do not intersect E. For the remaining balls, let $x_k \in A \cap B_{\delta/2}(z_k)$. Now $\{B_\delta(x_k)\}_{k=1,2,\ldots,N}$ covers $E \cap B_1$, $x_k \in E$, with $N \leq C\delta^{-m}$, and thus $N\delta^{m+\beta} \leq C\delta^\beta \leq 1/2$, provided that $\delta > 0$ is small enough.

Now observe that we can repeat this argument with $B_\delta(x_k)$ in place of B_1 to get a new covering $\{B_{\delta^2}(x_{k,l})\}_{l=1,2,\ldots,N_k}$ of $E \cap B_\delta(x_k)$ with $N_k\delta^{m+\beta} < 1/2$. Thus $\{B_{\delta^2}(x_{k,l})\}_{k=1,2,\ldots,N,\, l=1,2,\ldots,N_k}$ covers E with $x_{k,l} \in E$ and $\sum_{k=1}^N N_k\delta^{2\cdot(m+\beta)} < (1/2)^2$. Repeating this argument for a total of j times, we get a finite covering of E by M balls with centers on E, radii $= \delta^j$, and $M\delta^{j(m+\beta)} < (1/2)^j$. Thus $\mathcal{H}^{m+\beta}(E) \leq C(1/2)^j$ for every integer $j = 1, 2, 3, \ldots$. Letting $j \to \infty$, we get $\mathcal{H}^{m+\beta}(E) = 0$. □

Proof (Proof of Proposition 12) It follows from Lemma 9 and the definition of Hausdorff dimension. □

References

1. M. Allen, H. Shahgholian, *A new boundary Harnack principle (equations with right hand side)*, Arch. Rat. Mech. Anal., in press (2019).
2. I. Athanasopoulos, L. Caffarelli, *Optimal regularity of lower dimensional obstacle problems*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **310** (2004).
3. I. Athanasopoulos, L. Caffarelli, S. Salsa, *The structure of the free boundary for lower dimensional obstacle problems*, Amer. J. Math. **130** (2008), 485-498.
4. B. Barrios, A. Figalli, X. Ros-Oton, *Global regularity for the free boundary in the obstacle problem for the fractional Laplacian*, Amer. J. Math. **140** (2018), 415-447.
5. L. Caffarelli, *The regularity of free boundaries in higher dimensions*, Acta Math. **139** (1977), 155-184.

6. L. Caffarelli, *Further regularity for the Signorini problem*, Comm. Partial Differential Equations **4** (1979), 1067-1075.
7. L. Caffarelli, *The obstacle problem revisited*, J. Fourier Anal. Appl. **4** (1998), 383-402.
8. L. Caffarelli, S. Salsa, *A Geometric Approach to Free Boundary Problems*, American Mathematical Society, 2005.
9. M. Colombo, L. Spolaor, B. Velichkov, *A logarithmic epiperimetric inequality for the obstacle problem*, Geom. Funct. Anal. **28** (2018), 1029-1061.
10. M. Colombo, L. Spolaor, B. Velichkov, *Direct epiperimetric inequalities for the thin obstacle problem and applications*, Comm. Pure Appl. Math., to appear.
11. R. Cont, P. Tankov, *Financial modeling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.
12. E. De Giorgi, *Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari*, Memorie della Accademia delle Scienze di Torino. Classe di Scienze Fisiche, Matematicahe e Naturali **3** (1957), 25-43.
13. D. Danielli, N. Garofalo, A. Petrosyan, T. To, *Optimal regularity and the free boundary in the parabolic Signorini problem*, Mem. Amer. Math. Soc. **249** (2017), no. 1181, v + 103 pp.
14. D. De Silva, O. Savin, *A note on higher regularity boundary Harnack inequality*, Disc. Cont. Dyn. Syst. **35** (2015), 6155-6163.
15. D. De Silva, O. Savin, *Boundary Harnack estimates in slit domains and applications to thin free boundary problems*, Rev. Mat. Iberoam. **32** (2016), 891-912.
16. G. Duvaut, J. L. Lions, *Inequalities in Mechanics and Physics*, Grundlehren der Mathematischen Wissenschaften, Vol. **219**. Springer-Verlag, 1976.
17. X. Fernández-Real, Y. Jhaveri, *On the singular set in the thin obstacle problem: higher order blow-ups and the very thin obstacle problem*, preprint arXiv (2018).
18. X. Fernández-Real, X. Ros-Oton, *Regularity Theory for Elliptic PDE*, forthcoming book (2019).
19. A. Figalli, J. Serra, *On the fine structure of the free boundary for the classical obstacle problem*, Invent. Math. **215** (2019), 311-366.
20. M. Focardi, E. Spadaro, *On the measure and the structure of the free boundary of the lower dimensional obstacle problem*, Arch. Rat. Mech. Anal. **230** (2018), 125-184.
21. A. Friedman, *Variational Principles and Free Boundary Problems*, Dover, 1988.
22. N. Garofalo, A. Petrosyan, *Some new monotonicity formulas and the singular set in the lower dimensional obstacle problem*, Invent. Math. **177** (2009), 414-461.
23. N. Kikuchi, J. T. Oden, *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM Studies in Applied Mathematics, vol. **8**. Society for Industrial and Applied Mathematics, Philadelphia, 1988.
24. D. Kinderlehrer, L. Nirenberg, *Regularity in free boundary problems*, Ann. Sc. Norm. Sup. Pisa **4** (1977), 373-391.
25. D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, SIAM, 1980.
26. H. Koch, A. Petrosyan, W. Shi, *Higher regularity of the free boundary in the elliptic Signorini problem*, Nonlinear Anal. **126** (2015), 3-44.
27. B. Krummel, N. Wickramasekera, *Fine properties of branch point singularities: two-valued harmonic functions*, preprint arXiv (2013).
28. R. Merton, *Option pricing when the underlying stock returns are discontinuous*, J. Finan. Econ. **5** (1976), 125-144.
29. R. Monneau, *On the number of singularities for the obstacle problem in two dimensions*, J. Geom. Anal. **13** (2003), 359-389.
30. J. F. Nash, *Parabolic equations*, Proc. Nat. Acad. Sci. U.S.A. **43** (1957), 754-758.
31. J. F. Nash, *Continuity of solutions of parabolic and elliptic equations*, Amer. J. Math. **80** (1958), 931-954.
32. A. Petrosyan, H. Shahgholian, N. Uraltseva, *Regularity of free boundaries in obstacle-type problems*, Graduate Studies in Mathematics, Vol. **136**. American Mathematical Society, Providence, RI, 2012.

33. J.-F. Rodrigues, *Obstacle Problems in Mathematical Physics*, North-Holland mathematics Studies **134**, 1987.
34. X. Ros-Oton, *Obstacle problems and free boundaries: an overview*, SeMA J. **75** (2018), 399-419.
35. D. G. Schaeffer, *An example of generic regularity for a nonlinear elliptic equation*, Arch. Rat. Mech. Anal. **57** (1974), 134-141.
36. D. G. Schaeffer, *Some examples of singularities in a free boundary*, Ann. Scuola Norm. Sup. Pisa **4** (1977), 133-144.
37. W. Shi, *An epiperimetric inequality approach to the parabolic Signorini problem*, preprint arXiv (2018).
38. A. Signorini, *Sopra alcune questioni di elastostatica*, Atti Soc. It. Progr. Sc. **21** (1933), no. 2, 143-148.
39. A. Signorini, *Questioni di elasticità non linearizzata e semilinearizzata*, Rend. Mat. e Appl. **18** (1959), 95-139.
40. G. S. Weiss, *A homogeneity improvement approach to the obstacle problem*, Invent. Math. **138** (1999), 23-50.
41. B. White, Stratification of minimal surfaces, mean curvature flows, and harmonic maps, *J. Reine Angew. Math.* **488** (1997), 1-35.