Galois representations associated to ordinary Hilbert modular forms: Wiles' theorem

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Abstract

These are the notes of a talk given by the author at the STNB in January of 2015. It was the fourth talk in a series of five talks coordinated by Victor Rotger devoted to Galois representations attached to modular forms.

Contents

1 Introduction

The goal of this talk is to present Wiles' theorem on the existence of Galois representations associated to ordinary Hilbert modular forms of parallel weight $k \geq 1$ attached to a totally real number field F (see §2 for a precise statement), and to sketch the proof of the result.

The theorem follows from a result on the existence of Λ-adic representations attached to Λ-adic modular forms (due to Hida for $F = \mathbb{Q}$ and to Wiles in general; see Theorem 3.28), a lifting theorem of classical modular forms to Λ-adic modular forms (due to Hida for $k \geq 2$ and $F = \mathbb{Q}$, and to Wiles for $k \geq 1$

and general F; see Theorem 3.27), and the theorem of Carayol that we have seen in the third talk (see Theorem 2.3).

Wiles' proof of Theorem 3.28 relies on his theory of pseudo-representations, which we will also introduce. A funny aspect (which we will treat) of Wiles' method is that, in the case $F = \mathbb{Q}$, it permits to immediately deduce the results of Deligne/Deligne–Serre (seen in the first and the second talks) for weights $k \geq 2$ and $k = 1$ from the classical theory of Eichler-Shimura for weight $k = 2$.

In a subsequent paper, Taylor [Tay89] removed the ordinarity hypothesis establishing the existence of Galois representations associated to (non-necessarily ordinary) Hilbert modular forms attached to a totally real number field F of even degree $d := [F : \mathbb{Q}]$ and of (non-necessarily parallel) weight $k = (k_1, \ldots, k_d)$ with $k_j \geq 2$ for $j = 1, \ldots, d$.

We note that Lafferty's [Laf] presentation of Wiles' theorem has been useful at several passages.

2 General notations and statement of the result

Let F be a totally real field and $\mathcal{O}_F$ its ring of integers. Set $d := [F : \mathbb{Q}]$, and write h for its strict class number and $\mathfrak{d}$ for its different. Let $\{\mathfrak{t}_\gamma\}_{\gamma=1,\ldots,h}$ be a set of ideal representatives of the strict ideal classes of F. For $k \geq 1$, an integral ideal $\mathfrak{n}$ of $\mathcal{O}_F$, and a character $\psi_0 \colon (\mathcal{O}_F/\mathfrak{n})^* \to \overline{\mathbb{Q}}^*$, we denote by $S_k(\mathfrak{n}, \psi_0)$ the *space of cuspidal forms*

$$\mathbf{f} := (f_1, \ldots, f_h) \in \prod_{\gamma=1}^{h} S_k(\Gamma(\mathfrak{t}_\gamma \mathfrak{d}, \mathfrak{n}), \psi_0)\,.$$

Here, $f_\gamma \colon \mathbb{H}^d \to \mathbb{C}$, for $\gamma \in \{1, \ldots, h\}$, is a Hilbert cuspidal form of parallel weight $k \geq 1$, character ψ_0, and level

$$\Gamma(\mathfrak{t}_\gamma \mathfrak{d}, \mathfrak{n}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(F) \,|\, a, d \in \mathcal{O}_F^*, b \in t_\gamma^{-1}\mathfrak{d}^{-1}, c \in \mathfrak{t}_\gamma \mathfrak{d}\mathfrak{n}, ad - bc \in \mathcal{O}_F^* \right\}$$

in the sense of the third talk. Here, $\mathbb{H}$ denotes Poincaré upper half plane.

We saw that, for each $\gamma \in \{1, \ldots, h\}$, we can represent f_γ by its Fourier expansion

$$f_\gamma(z_1, \ldots, z_d) = \sum_{0 \ll \mu \in \mathfrak{t}_\gamma} a_\gamma(\mu) e^{2\pi i (\sum_{j=1}^{d} \mu_j z_j)}\,, \quad \text{for } (z_1, \ldots, z_d) \in \mathbb{H}^d.$$

In the sum, μ runs over totally positive elements of the lattice $\mathfrak{t}_\gamma$, and $\mu_1, \ldots, \mu_d$ denote the images of μ by the d distinct embeddings of F into $\mathbb{C}$.

Let $\mathfrak{a} \subseteq \mathcal{O}_F$ be a nonzero integral ideal. Then (by definition) there exist $\gamma \in \{1, \ldots, h\}$ and a totally positive $\mu \in \mathfrak{t}_\gamma$ such that $\mathfrak{a} = \mu\mathfrak{t}_\gamma^{-1}$. For a fractional ideal $\mathfrak{a}$ of F, the numbers

$$c(\mathfrak{a}, \mathbf{f}) := \begin{cases} a_\gamma(\mu) N(\mathfrak{t}_\gamma)^{-k/2} & \text{if } \mathfrak{a} = (\mu)\mathfrak{t}_\gamma^{-1} \text{ is integral,} \\ 0 & \text{otherwise,} \end{cases}$$

depend neither on the choice of the $\mathfrak{t}_\gamma$'s nor on the choice of μ. The *Dirichlet series* associated to the cuspidal form $\mathbf{f}$ is then given by

$$D(\mathbf{f}, s) := \sum_{0 \neq \mathfrak{a} \subseteq \mathcal{O}_F} c(\mathfrak{a}, \mathbf{f}) N(\mathfrak{a})^{-s} .$$

There is a theory of Hecke operators on $S_k(\mathfrak{n}, \psi_0)$ given by $\{T_\mathfrak{n}(\mathfrak{a}), S_\mathfrak{n}(\mathfrak{a})\}$, where $\mathfrak{a} \subseteq \mathcal{O}_F$ is an integral ideal (see [Shi78, §2]). Let $\psi \colon I_{\mathfrak{n}\infty} \to \overline{\mathbb{Q}}^*$ be a ray class character of modulus $\mathfrak{n}\infty$, where ∞ is the product of the infinite places of F, that restricts to ψ_0 on $(\mathcal{O}_F/\mathfrak{n})^*$. Then set[1]

$$S_k(\mathfrak{n}, \psi) := \{\mathbf{f} \in S_k(\mathfrak{n}, \psi_0) \mid S_\mathfrak{n}(\mathfrak{a})(\mathbf{f}) = \psi(\mathfrak{a})\mathbf{f} \text{ for all } \mathfrak{a} \subseteq \mathcal{O}_F\} .$$

Suppose from now on that $\mathbf{f} \in S_k(\mathfrak{n}, \psi)$ is a newform (that is, it is normalized meaning that $c(\mathcal{O}_F, \mathbf{f}) = 1$, it is new at level $\mathfrak{n}$, and $T_\mathfrak{n}(\mathfrak{a})\mathbf{f} = c(\mathfrak{a}, \mathbf{f})\mathbf{f}$ for every integral ideal $\mathfrak{a} \subseteq \mathcal{O}_F$). Let $K_\mathbf{f}$ denote the number field generated by the set of eigenvalues $\{c(\mathfrak{a}, \mathbf{f})\}_{\mathfrak{a} \subseteq \mathcal{O}_F}$ and denote by $\mathcal{O}_\mathbf{f}$ its ring of integers. Let λ be a prime of $\mathcal{O}_\mathbf{f}$ and denote by $\mathcal{O}_{\mathbf{f},\lambda}$ the completion of $\mathcal{O}_\mathbf{f}$ at λ.

Definition 2.1. We say that $\mathbf{f}$ is *ordinary* at λ if for each prime $\mathfrak{p} \subseteq \mathcal{O}_F$ dividing the norm $N(\lambda)$ the equation

$$x^2 - c(\mathfrak{p}, \mathbf{f})x + \psi(\mathfrak{p})N(\mathfrak{p})^{k-1}$$

has at least one root which is a unit mod λ.

Let us write G_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$.

Theorem 2.2. *[Wil88, Thm. 1] Let $\mathbf{f} \in S_k(\mathfrak{n}, \psi)$ be a newform with $k \geq 1$. If $\mathbf{f}$ is ordinary at λ, there exists a continuous irreducible representation*

$$\varrho_{\mathbf{f},\lambda} \colon G_F \to \mathrm{GL}_2(\mathcal{O}_{\mathbf{f},\lambda}) \tag{2.1}$$

unramified outside $\mathfrak{n}N(\lambda)$ and such that, for all prime ideals $\mathfrak{q} \nmid \mathfrak{n}N(\lambda)$, one has

$$\mathrm{Tr}(\varrho_{\mathbf{f},\lambda})(\mathrm{Frob}_\mathfrak{q}) = c(\mathfrak{q}, \mathbf{f}) ,$$
$$\det(\varrho_{\mathbf{f},\lambda})(\mathrm{Frob}_\mathfrak{q}) = \psi(\mathfrak{q})N(\mathfrak{q})^{k-1} .$$

In the third talk, we saw the following result of Carayol. It will be a fundamental tool in the proof of the above theorem.

Theorem 2.3. *[Car86, Thm. (B)] For a newform $\mathbf{f} \in S_k(\mathfrak{n}, \psi)$ with $k \geq 2$ (not necessarily ordinary at λ), there exists a representation as in (2.1) if either*

i) $d := [F : \mathbb{Q}]$ is odd; or

ii) d is even and there is a prime ideal $\mathfrak{p}$ dividing exactly $\mathfrak{n}$ which does not divide the conductor of ψ.

[1] There is no distinction between $S_k(\mathfrak{n}, \psi_0)$ and $S_k(\mathfrak{n}, \psi)$ for $F = \mathbb{Q}$. Indeed, for $N \geq 1$ the ray class group of modulus $(N)\infty$ is isomorphic to $(\mathbb{Z}/N\mathbb{Z})^*$.

In fact, the above is a weaker statement than the one in Carayol's theorem, but it will suffice for deducing Theorem 2.2. One could relax *ii*), by just requiring that for some prime ideal $\mathfrak{p}$ dividing $\mathfrak{n}$, we have that $\mathbf{f}$ is special or supercuspidal locally at $\mathfrak{p}$. One can indeed show that if *ii*) holds, then $\mathbf{f}$ is unramified special at $\mathfrak{p}$ (see Lemma 4.6).

The easiest case covered by Theorem 2.2 and not by Theorem 2.3 corresponds to taking $\mathbf{f}$ attached to a real quadratic field F and with trivial level $\mathfrak{n} = \mathcal{O}_F$.

3 Main notions for the proof

We introduce the three main notions required for the proof of Theorem 2.2: pseudo-representations, p-stabilized modular forms, and Λ-adic modular forms.

3.1 Pseudo-representations

The first ingredient in the proof of Theorem 2.2 is the notion of pseudo-representation.

Definition 3.1. Let G be a profinite group and let R be a commutative topological integral domain (with unity). A *pseudo-representation* of G into R is a triple $\pi = (A_\pi, D_\pi, C_\pi)$ of continuous maps

$$A_\pi \colon G \to R\,, \quad D_\pi \colon G \to R\,, \quad C_\pi \colon G \times G \to R$$

satisfying the following conditions for all elements $g, g_i \in G$:

i) $A_\pi(g_1 g_2) = A_\pi(g_1) A_\pi(g_2) + C_\pi(g_1, g_2)$.

ii) $D_\pi(g_1 g_2) = D_\pi(g_1) D_\pi(g_2) + C_\pi(g_1, g_2)$.

iii) $C_\pi(g_1 g_2, g_3) = A_\pi(g_1) C_\pi(g_2, g_3) + D_\pi(g_2) C_\pi(g_1, g_3)$.

iv) $C_\pi(g_1, g_2 g_3) = A_\pi(g_3) C_\pi(g_1, g_2) + D_\pi(g_2) C_\pi(g_1, g_3)$.

v) $A_\pi(1) = D_\pi(1) = 1$.

vi) $C_\pi(g, 1) = C_\pi(1, g) = 0$.

vii) $C_\pi(g_1, g_2) C_\pi(g_3, g_4) = C_\pi(g_1, g_4) C_\pi(g_3, g_2)$.

Remark 3.2. Note that C_π is determined by both A_π and D_π (as follows from i) and ii)). Its consideration responds to merely notational purposes.

Lemma 3.3. *One has:*

- *If* $\varrho \colon G \to \mathrm{GL}_2(R)$ *is a representation with*

  $$\varrho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}\,,$$

  then $\pi_\rho := (A, D, C)$ *with*

  $$A(g) := a(g), \quad D(g) := d(g), \quad C(g_1, g_2) := b(g_1) c(g_2)$$

  defines a pseudo-representation.

- *Conversely, if $\pi = (A, D, C)$ is a pseudo-representation of G into R such that $C = 0$ (resp. such that there exist $g_1, g_2 \in G$ with $C(g_1, g_2) \in R^*$), then*

$$\varrho_\pi(g) := \begin{pmatrix} A(g) & 0 \\ 0 & D(g) \end{pmatrix} \qquad \left(\textit{resp.} \quad \varrho_\pi(g) := \begin{pmatrix} A(g) & C(g, g_2)/C(g_1, g_2) \\ C(g_1, g) & D(g) \end{pmatrix} \right)$$

defines a representation $\varrho_\pi \colon G \to \mathrm{GL}_2(R)$.

Remark 3.4. The notion of pseudo-representation makes precise the naïve idea that a representation should consist of a tuple of functions $G \to R$ satisfying a series of compatibility relations. To illustrate that the set of compatibility conditions in Definition 3.1 is the right one, let us prove that the map

$$\varrho_\pi \colon G \to \mathrm{GL}_2(R)$$

defined in Lemma 3.3 from a pseudo-representation $\pi = (A, D, C)$ is indeed a homomorphism. We will just consider the interesting case in which there exist $g_1, g_2 \in G$ with $C(g_1, g_2) \in R^*$, the other being obvious. Indeed,

$$\begin{aligned}
\varrho_\pi(h_1 h_2) &= \begin{pmatrix} A(h_1 h_2) & C(h_1 h_2, g_2)/C(g_1, g_2) \\ C(g_1, h_1 h_2) & D(h_1 h_2) \end{pmatrix} \\
&\overset{i),\dots,iv)}{=} \begin{pmatrix} A(h_1)A(h_2) + C(h_1, h_2) & \frac{A(h_1)C(h_2,g_2)+D(h_2)C(h_1,g_2)}{C(g_1,g_2)} \\ A(h_2)C(g_1, h_1) + D(h_1)C(g_1, h_2) & D(h_1)D(h_2) + C(h_2, h_1) \end{pmatrix} \\
&\overset{vii)}{=} \begin{pmatrix} A(h_1)A(h_2) + \frac{C(h_1,g_2)C(g_1,h_2)}{C(g_1,g_2)} & \frac{A(h_1)C(h_2,g_2)+D(h_2)C(h_1,g_2)}{C(g_1,g_2)} \\ A(h_2)C(g_1, h_1) + D(h_1)C(g_1, h_2) & D(h_1)D(h_2) + \frac{C(h_2,g_2)C(g_1,h_1)}{C(g_1,g_2)} \end{pmatrix} \\
&= \begin{pmatrix} A(h_1) & C(h_1, g_2)/C(g_1, g_2) \\ C(g_1, h_1) & D(h_1) \end{pmatrix} \begin{pmatrix} A(h_2) & C(h_2, g_2)/C(g_1, g_2) \\ C(g_1, h_2) & D(h_2) \end{pmatrix} \\
&= \varrho_\pi(h_1)\varrho_\pi(h_2)
\end{aligned}$$

In particular, as a consequence of the previous lemma, one has that if R is field, then every pseudo-representation in R comes from a representation with values in $\mathrm{GL}_2(R)$. In view of the previous lemma, the following definition is natural.

Definition 3.5. The *trace* and *determinant* of a pseudo-representation $\pi = (A_\pi, D_\pi, C_\pi)$ of G into R are defined by

$$\mathrm{Tr}(\pi)(g) := A_\pi(g) + D_\pi(g)\,, \qquad \det(\pi)(g) := A_\pi(g)D_\pi(g) - C_\pi(g, g)\,.$$

Remark 3.6. Let π_ϱ be the pseudo-representation attached to a representation ϱ. Then

$$\mathrm{Tr}(\pi_\varrho) = \mathrm{Tr}(\varrho), \qquad \det(\pi_\varrho) = \det(\varrho)\,.$$

Recall that a representation $\varrho \colon G \to \mathrm{GL}_2(R)$ is called *odd* if there exists $c \in G$ of order 2, such that

$$\varrho(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition 3.7. We say that $\pi = (A_\pi, D_\pi, C_\pi)$ is an *odd* pseudo-representation if there exists $c \in G$ of order 2 such that, for every $g \in G$, we have that

$$A_\pi(c) = -1, \qquad D_\pi(c) = 1, \qquad C_\pi(g, c) = 0, \qquad C_\pi(c, g) = 0\,.$$

Lemma 3.8. *If 2 is invertible in R, then an odd pseudo-representation π is determined by* $\mathrm{Tr}(\pi)$*.*

Proof. Indeed:

$$\begin{aligned}
A_\pi(g) &= \tfrac{1}{2}\left(A_\pi(g) + D_\pi(g) - (D_\pi(g) - A_\pi(g))\right) = \\
&= \tfrac{1}{2}(A_\pi(g) + D_\pi(g) - (D_\pi(gc) + A_\pi(gc))) = \tfrac{1}{2}(\mathrm{Tr}(\pi)(g) - \mathrm{Tr}(\pi)(gc))\,, \\
D_\pi(g) &= \tfrac{1}{2}(A_\pi(g) + D_\pi(g) + (D_\pi(g) - A_\pi(g))) = \\
&= \tfrac{1}{2}(A_\pi(g) + D_\pi(g) + (D_\pi(gc) + A_\pi(gc))) = \tfrac{1}{2}(\mathrm{Tr}(\pi)(g) + \mathrm{Tr}(\pi)(gc))\,, \\
C_\pi(g_1, g_2) &= A_\pi(g_1 g_2) - A_\pi(g_1)A_\pi(g_2)\,.
\end{aligned}$$

□

Observe that by the previous lemma, if R is a field of characteristic 0, then there is a 1-1 correspondence between odd semisimple representations into $\mathrm{GL}_2(R)$ and odd pseudo-representations in R.

Remark 3.9. We fix from now on an algebraic closure of the fraction field $\mathbb{Q}_p((X))$ of $\mathbb{Z}_p[[X]]$, where p is a prime. Any algebraic extension of $\mathbb{Q}_p((X))$ is assumed to be contained in this fixed algebraic closure. Let $\mathcal{K}$ denote a finite algebraic extension of $\mathbb{Q}_p((X))$ and let Λ denote the integral closure of $\mathbb{Z}_p[[X]]$ in $\mathcal{K}$. We will be concerned with Galois pseudo-representations of G_F into $R = \Lambda$.

Remark 3.10. There are two types of prime ideals P of height 1 in Λ. On the one hand, we have those P lying over p. There are only a finite number of them, and in this case Λ/P is a finite extension of $\mathbb{F}_p[[X]]$. On the other hand, we have those P not dividing p. In this case, Λ/P is a finite extension of $\mathbb{Z}_p$.

Theorem 3.11. *[Wil88, Lem. 2.2.3] Let $\{P_n\}_{n=1}^{\infty}$ be a sequence of distinct height 1 prime ideals of Λ. Let K_n denote the field of fractions of Λ/P_n, and let $\mathcal{O}_n$ be the integral closure of Λ/P_n in K_n. Suppose that for each $n \geq 1$, there exists a continuous odd representation*

$$\varrho_n \colon G_F \to \mathrm{GL}_2(\mathcal{O}_n)$$

that is unramified outside $\mathfrak{n}p$, for some integral ideal $\mathfrak{n} \subseteq \mathcal{O}_F$. Furthermore, suppose that for every prime $\mathfrak{q} \nmid \mathfrak{n}p$, there exist $c_\mathfrak{q}(X), \varepsilon_\mathfrak{q}(X) \in \Lambda$ such that

$$\begin{aligned}
\mathrm{Tr}(\varrho_n)(\mathrm{Frob}_\mathfrak{q}) &\equiv c_\mathfrak{q}(X) \pmod{P_n}\,, \\
\det(\varrho_n)(\mathrm{Frob}_\mathfrak{q}) &\equiv \varepsilon_\mathfrak{q}(X) \pmod{P_n}\,.
\end{aligned}$$

Then there exists a continuous odd representation $\varrho\colon G_F \to \mathrm{GL}_2(\mathcal{K})$ unramified outside $\mathfrak{n}p$ and such that

$$\begin{aligned}\mathrm{Tr}(\varrho)(\mathrm{Frob}_{\mathfrak{q}}) &= c_{\mathfrak{q}}(X)\,,\\ \det(\varrho)(\mathrm{Frob}_{\mathfrak{q}}) &= \varepsilon_{\mathfrak{q}}(X)\,,\end{aligned}$$

for every prime $\mathfrak{q} \nmid \mathfrak{n}p$. Furthermore, ϱ is absolutely irreducible if and only if ϱ_n is for some n.

Remark 3.12. The notion of continuity for a Galois representation $\varrho\colon G_F \to \mathrm{GL}_2(\mathcal{K})$ on a 2-dimensional $\mathcal{K}$-vector space is not relative to the topology of $\mathrm{GL}_2(\mathcal{K})$ as a subspace of $\mathcal{K} \times \mathcal{K} \times \mathcal{K} \times \mathcal{K}$. Before proceeding to the proof of Theorem 3.11, let us describe the notion of continuity that is used in its statement. Recall that the ring Λ is complete, local, Noetherian, and of Krull dimension 2. Let $\mathfrak{m}$ denote its maximal ideal. A *lattice* of $\mathcal{K}^2$ is a sub-Λ-module L of $\mathcal{K}^2$ of finite type over Λ such that $L \otimes_\Lambda \mathcal{K} = \mathcal{K}^2$. We say that ϱ is *continuous* if there exists a lattice L of $\mathcal{K}^2$ that is stable under ϱ and such that

$$\varrho\colon G_F \to \mathrm{Aut}_\Lambda(L)$$

is continuous with respect to the projective limit topology (=Krull topology) on

$$\mathrm{Aut}_\Lambda(L) \simeq \varprojlim \mathrm{Aut}(L/\mathfrak{m}^j L)\,.$$

Proof of Theorem 3.11. First observe that by hypothesis and the Cebotarev density Theorem, $\mathrm{Tr}(\varrho_n)$ takes values in Λ/P_n. Let π_n be the pseudo-representation with values (in principle) in $\mathcal{O}_n$ attached by Lemma 3.3 to the representation ϱ_n. Note that $\mathrm{Tr}(\pi_n)$ coincides with $\mathrm{Tr}(\varrho_n)$ by Remark 3.6, and thus it takes values in Λ/P_n. But since π_n is odd (as ϱ_n is), it is determined by $\mathrm{Tr}(\pi_n)$ as in the proof of Lemma 3.8 and it takes values in the same ring as $\mathrm{Tr}(\pi_n)$, that is, π_n takes values in Λ/P_n (this is one of the key points of considering pseudo-representations!). Let us write $Q_r = P_1 \cap \cdots \cap P_r$. Suppose that we have constructed a pseudo-representation α_r in Λ/Q_r such that $\alpha_r \equiv \pi_n \pmod{P_n}$ for $1 \leq n \leq r$ (to start the induction process, just take $\alpha_1 := \pi_1$). By the hypothesis (together with the Cebotarev density theorem), we have that for $1 \leq n \leq r$

$$\mathrm{Tr}(\alpha_r) \equiv \mathrm{Tr}(\pi_n) = \mathrm{Tr}(\varrho_n) \equiv \mathrm{Tr}(\varrho_{r+1}) = \mathrm{Tr}(\pi_{r+1}) \pmod{(P_n, P_{r+1})}.$$

This implies

$$\mathrm{Tr}(\alpha_r) \equiv \mathrm{Tr}(\pi_{r+1}) \pmod{(Q_r, P_{r+1})}.$$

But by Lemma 3.8, an odd pseudo-representation is determined by its trace, and thus

$$\alpha_r \equiv \pi_{r+1} \pmod{(Q_r, P_{r+1})}\,.$$

Thanks to this congruence and using the exact sequence

$$0 \to \Lambda/Q_{r+1} \to \Lambda/Q_r \oplus \Lambda/P_{r+1} \to \Lambda/(Q_r, P_{r+1}) \to 0\,, \tag{3.1}$$

we may lift the pseudo-representation $\alpha_r \oplus \pi_{r+1}$ of G_F into $\Lambda/Q_r \oplus \Lambda/P_{r+1}$ to a pseudo-representation α_{r+1} of G_F into Λ/Q_{r+1}, with the property that $\alpha_{r+1} \equiv \pi_n \pmod{P_n}$ for $1 \leq n \leq r+1$. Then $\lim_{\leftarrow} \alpha_n$ is a pseudo-representation of G_F into $\lim_{\leftarrow} \Lambda/P_n$. This projective limit is canonically isomorphic to Λ, since by hypothesis $\{P_n\}_{n=1}^{\infty}$ is an *infinite* set of distinct height 1 prime ideals and thus whose intersection is 0. By Lemma 3.3, taking coefficients over $\mathcal{K}$, the pseudo-representation $\lim_{\leftarrow} \alpha_n$ defines a representation of G_F, which has the desired properties. $\square$

Remark 3.13. As already mentioned, a key point of the proof is the following: even when the representation ϱ_n of the theorem is has coefficients in the integral closure $\mathcal{O}_n$ of Λ/P_n, the attached pseudo-representation π_n takes values in Λ/P_n. In the process of "patching together" the π_n, we make use of the exact sequence (3.1), which we have for the rings Λ/P_n (without having to worry about their integral closures).

3.2 The space of p-stabilized modular forms

For a subring A of $\mathbb{C}$, define

$$S_k(\mathfrak{n}, \psi \,|\, A) := \{\mathbf{g} \in S_k(\mathfrak{n}, \psi) \,|\, c(\mathfrak{a}, \mathbf{g}) \in A \text{ for all } \mathfrak{a} \subseteq \mathcal{O}_F\}\,. \tag{3.2}$$

Fix an algebraic closure $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}}_p$) of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) and let $\mathbb{C}_p$ denote the completion of $\overline{\mathbb{Q}}_p$ with respect to the normalized absolute value. Fix an embedding $i_p \colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_p$. Without any further word, any algebraic extension of $\mathbb{Q}$ (resp. $\mathbb{Q}_p$) that we consider will be assumed to belong to $\overline{\mathbb{Q}}$ (resp. $\overline{\mathbb{Q}}_p$).

The space $S_k(\mathfrak{n}, \psi)$ has a basis with coefficients in $\mathbb{Z}[\psi]$. For any subring $\mathbb{Z}_p[\psi] \subseteq \mathcal{O} \subseteq \mathbb{C}_p$, we thus define

$$S_k(\mathfrak{n}, \psi \,|\, \mathcal{O}) := S_k(\mathfrak{n}, \psi \,|\, \mathbb{Z}[\psi]) \otimes_{\mathbb{Z}[\psi]} \mathcal{O}\,.$$

We assume until the end of this section that the level is of the form $\mathfrak{n}p^r$ for $r \geq 1$ and that $\mathcal{O}$ is a finite extension of $\mathbb{Z}_p$ containing $\mathbb{Z}[\psi]$. Since the level is understood, we will simply write $T(p)$ for the Hecke operator $T_{\mathfrak{n}p^r}(p)$. The *Hida operator* is defined by

$$e := \lim_{n \to \infty} T(p)^{n!} \colon S_k(\mathfrak{n}p^r, \psi \,|\, \mathcal{O}) \to S_k(\mathfrak{n}p^r, \psi \,|\, \mathcal{O})\,.$$

One can show that e is well-defined and an idempotent of $\mathrm{End}_{\mathcal{O}}(S_k(\mathfrak{n}p^r, \psi \,|\, \mathcal{O}))$. We define the space of *p-stabilized cuspidal forms* by

$$S_k^{\mathrm{ord}}(\mathfrak{n}p^r, \psi \,|\, \mathcal{O}) := eS_k(\mathfrak{n}p^r, \psi \,|\, \mathcal{O}).$$

Remark 3.14. Let $\mathbf{f} \in S_k(\mathfrak{n}p^r, \psi \,|\, \mathcal{O})$ be a newform of level $\mathfrak{m}|\mathfrak{n}p^r$ and weight $k \geq 2$. Then $e\mathbf{f}$ is nonzero if and only if $\mathbf{f}$ is ordinary[2] (i.e. $c(p, \mathbf{f})$ is a unit in $\mathcal{O}$).

[2]This somehow justifies the notation $S_k^{\mathrm{ord}}(\mathfrak{n}p^r, \psi \,|\, \mathcal{O})$. Note however that not every ordinary cuspidal form lies in $S_k^{\mathrm{ord}}(\mathfrak{n}p^r, \psi \,|\, \mathcal{O})$. It is rather the p-stabilization of any ordinary cuspidal form that lies in $S_k^{\mathrm{ord}}(\mathfrak{n}p^r, \psi \,|\, \mathcal{O})$.

In this case, $e\mathbf{f}$ is a newform of level $\mathfrak{m}\mathfrak{P}$, where $\mathfrak{P}$ is the product of primes above p which do not divide $\mathfrak{m}$. The eigenvalue of $e\mathbf{f}$ for $\mathfrak{q} \nmid \mathfrak{P}$ is the same as for $\mathbf{f}$; the eigenvalue of $e\mathbf{f}$ for $\mathfrak{q}|\mathfrak{P}$ is the unit root of $x^2 - c(\mathfrak{q}, \mathbf{f})x + \psi(\mathfrak{q})N(\mathfrak{q})^{k-1}$.

Remark 3.15. Suppose that $\mathbf{f} \in S_k(\mathfrak{n}p^r, \psi \,|\, \mathcal{O})$ is a newform of level $\mathfrak{m}|\mathfrak{n}p^r$, that $\mathfrak{q}|p^r$, and that $\mathfrak{q}\mathfrak{m}|\mathfrak{n}p^r$. Then $e\mathbf{f}(\mathfrak{q}\cdot)$ lies in the linear span of $e\mathbf{f}$. We deduce that $S_k^{\mathrm{ord}}(\mathfrak{n}p^r, \psi \,|\, \mathcal{O})$ is spanned by the set

$$\{e\mathbf{f}_i(\mathfrak{q}_i\cdot) \,|\, \mathbf{f}_i \text{ is a newform of level } \mathfrak{m}_i, \, (\mathfrak{q}_i, p) = 1, \mathfrak{m}_i\mathfrak{q}_i|\mathfrak{n}p^r\} \,.$$

We illustrate the previous two remarks with an example.

Example 3.16. Let $F = \mathbb{Q}$ and $N \geq 1$ with $(N, p) = 1$. Let $f = \sum_{n\geq 1} c_n q^n \in S_k(\Gamma_0(N), \psi)$ be an ordinary newform. Let α and β denote the roots of $x^2 - c_p(f)X + \psi(p)p^{k-1}$ and suppose that α is a unit.

The action of $T_{Np}(p)$ on q-expansions $g(q) = \sum_{n\geq 1} a_n q^n$ is well-known: by definition, one has that $T_{Np}(p)(g) = \sum_{n\geq 1} a_{np}q^n$. Recall that

$$S_{Np}(p)\colon S_k(\Gamma_0(Np), \psi) \to S_k(\Gamma_0(Np), \psi) \,, \qquad S_{Np}(p)(g) := g(q^p) \,.$$

One easily checks that $T_{Np}(p)$ stabilizes the 2-dimensional subspace generated by f and $S_{Np}(p)(f)$ of $S_2(\Gamma_0(Np), \psi)$. In this basis, we have

$$T_{Np}(p) = \begin{pmatrix} c_p & 1 \\ -\psi(p)p^{k-1} & 0 \end{pmatrix} .$$

The second column follows from the relation $T_{Np}(p) \circ S_{Np}(p) = id$ and the first is due to the relation $T_N(p) = T_{Np}(p) + \psi(p)p^{k-1}S_{Np}(p)$ together with the fact that $T_N(p)(f) = c_p f$.

One readily checks that $f_\alpha(z) := f(z) - \beta f(pz)$ and $f_\beta(z) := f(z) - \alpha f(pz)$ are the eigenvectors of the matrix attached to $T_{Np}(p)$ (of eigenvalues α and β, respectively). From the equalities

$$T_{Np}(p)(f_\alpha) = \alpha f_\alpha \,, \qquad T_{Np}(p)(f_\beta) = \beta f_\beta,$$

it follows that $ef_\alpha(z) = f_\alpha(z)$ and that $ef_\beta(z) = 0$. Solving the resulting two equations linear system, one deduces that

$$ef(z) = \frac{\alpha}{\alpha - \beta} f_\alpha(z) \,, \qquad e(f(pz)) = \frac{1}{\alpha - \beta} f_\alpha(z) \,.$$

3.3 Λ-adic modular forms

Hypothesis 3.17. *For simplicity, we assume from now on that p is a prime* ≥ 3.

Let $\mathbb{Q}_\infty$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$ and let $p^e = [F \cap \mathbb{Q}_\infty \colon \mathbb{Q}]$. Set $u := (1+p)^{p^e}$. For each $r \geq 0$, we fix a root of unity ζ of order p^r. For $k \geq 1$ and $r \geq 0$, define the *specialization map*

$$\nu_{k,r}\colon \mathbb{Z}_p[[X]] \to \mathbb{Z}_p[\zeta] \,, \qquad X \mapsto \zeta u^{k-2} - 1 \,.$$

Note that we may use $\nu_{k,r}$ to view $\mathbb{Z}_p[[X]]/\ker(\nu_{k,r})$ as a finite extension of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}}_p$. Since $\mathbb{Z}_p[[X]]$ has no zero divisors, $\ker(\nu_{k,r})$ is a prime ideal. It is the prime ideal generated by the minimal polynomial of $\zeta u^{k-2} - 1$ over Λ.

Remark 3.18. As in Remark 3.9, let $\mathcal{K}$ denote a finite extension of the fraction field $\mathbb{Q}_p((X))$ of $\mathbb{Z}_p[[X]]$, and let Λ denote the integral closure of $\mathbb{Z}_p[[X]]$ in $\mathcal{K}$. Suppose that $\mathcal{K}$ and Λ are large enough so that $\mathbb{Z}_p[\psi][[X]] \subseteq \Lambda$. Since Λ is integral and finitely generated over $\mathbb{Z}_p[[X]]$, by the *Going-up theorem* of Cohen-Seidenberg there exists a prime ideal $P_{k,r} \subseteq \Lambda$ such that $P_{k,r} \cap \mathbb{Z}_p[[X]] = \ker(\nu_{k,r})$. We thus have a diagram

$$\begin{array}{ccccc} P_{k,r} & \subseteq & \Lambda & \subseteq & \mathcal{K} \\ | & & | & & | \\ \ker(\nu_{k,r}) & \subseteq & \mathbb{Z}_p[[X]] & \subseteq & \mathbb{Q}_p((X)) \end{array}$$

Let $\mathcal{O} := \Lambda \cap \overline{\mathbb{Q}}_p$ and $K := \mathcal{K} \cap \overline{\mathbb{Q}}_p$, so that $\mathcal{O}$ is the valuation ring of the finite extension K of $\mathbb{Z}_p$ and $\mathbb{Z}_p[\psi] \subseteq \mathcal{O}$. The natural projection

$$\nu\colon \Lambda \to \Lambda/P_{k,r} \subseteq \overline{\mathbb{Q}}_p$$

is an $\mathcal{O}$-algebra homomorphism that extends $\nu_{k,r}$. However, ν depends on the choice of $P_{k,r}$ above $\ker(\nu_{k,r})$. Let $\mathfrak{X}_{k,r}$ denote the set of all $\mathcal{O}$-algebra homomorphisms from Λ to $\overline{\mathbb{Q}}_p$ that restrict to $\nu_{k,r}$ on $\mathbb{Z}_p[[X]]$. Write

$$\mathfrak{X} := \bigcup_{r\geq 0,\, k\geq 1} \mathfrak{X}_{k,r}\,.$$

Recall that for a fractional ideal $\mathfrak{a}$ of F such that $(\mathfrak{a}, p) = 1$, we can write

$$N(\mathfrak{a}) = u^{\alpha}\delta\,, \qquad \text{with } \delta \in \mu_{p-1},\ \alpha \in \mathbb{Z}_p\,.$$

Let $\psi\colon I_{\mathfrak{n}\infty} \to \overline{\mathbb{Q}}^*$ be as in §2. We define the following three characters

$$\begin{aligned} &\boldsymbol{\psi}\colon \varprojlim_t I_{\mathfrak{n}p^t} \to \Lambda\,, && \boldsymbol{\psi}(\mathfrak{a}) = \psi(\mathfrak{a})(1+X)^{\alpha}\,, \\ &\varrho_r\colon I_{p^r\mathcal{O}_F} \to \overline{\mathbb{Q}}^*\,, && \varrho_r(\mathfrak{a}) := \zeta^{\alpha}\,, \\ &\omega\colon I_{p\mathcal{O}_F} \to \overline{\mathbb{Q}}^*\,, && \omega(\mathfrak{a}) = N(\mathfrak{a})/u^{\alpha} = \delta\,. \end{aligned} \tag{3.3}$$

We will call ω the *Teichmüller character*.

Definition 3.19. A Λ-*adic cuspidal form* $\mathcal{F}$ over F of level $\mathfrak{n}$ and character $\boldsymbol{\psi}\colon \varprojlim_t I_{\mathfrak{n}p^t} \to \Lambda$ is a collection of elements of Λ

$$\{c(\mathfrak{a},\mathcal{F})(X)\}_{0\neq\mathfrak{a}\subseteq\mathcal{O}_F} \subseteq \Lambda\,,$$

with the property that, for all but finitely many $k \geq 2$ and $r \geq 0$ and for all $\nu \in \mathfrak{X}_{k,r}$, there exists

$$\mathbf{f}_\nu \in S_k(\mathfrak{n}p^r, \psi\varrho_r\omega^{2-k} \,|\, \mathcal{O}[\zeta])$$

whose associated Dirichlet series is

$$D(\mathbf{f}_\nu, s) = \sum_{0\neq\mathfrak{a}\subseteq\mathcal{O}_F} \nu(c(\mathfrak{a},\mathcal{F})(X))N(\mathfrak{a})^{-s}\,.$$

By abuse of notation, we will write $\nu(\mathcal{F}) = \mathbf{f}_\nu$.

Definition 3.20. We denote by $\mathcal{S}(\mathfrak{n}, \boldsymbol{\psi}\,|\,\Lambda))$ the *space of* Λ*-adic cuspidal forms* of level $\mathfrak{n}$ and character $\boldsymbol{\psi}$. Set

$$\mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda) = \bigcup_{t=0}^{\infty} \mathcal{S}(\mathfrak{n}p^t, \boldsymbol{\psi}\,|\,\Lambda)\,.$$

Remark 3.21. The specialization $\nu(\boldsymbol{\psi})$ and the central character ψ_ν of $\mathbf{f}_\nu = \nu(\mathcal{F})$ are related by the formula $N^{2-k}\nu(\boldsymbol{\psi}) = \psi_\nu$. Indeed, suppose that $\nu \in \mathfrak{X}_{k,r}$, so that $\psi_\nu = \psi\varrho_r\omega^{2-k}$. Then

$$\begin{aligned} N(\mathfrak{a})^{2-k}\nu(\boldsymbol{\psi}(\mathfrak{a})) &= N^{2-k}(\mathfrak{a})\psi(\mathfrak{a})\nu(1+X)^{\alpha} = \\ &= N^{2-k}(\mathfrak{a})\psi(\mathfrak{a})\zeta^{\alpha}u^{(k-2)\alpha} = \\ &= \psi(\mathfrak{a})\varrho_r(\mathfrak{a})\omega(\mathfrak{a})^{2-k}\,, \end{aligned}$$

from which the desired equality follows.

Remark 3.22. There exists an idempotent

$$\mathcal{E}\colon\ \mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda) \to \mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda)$$

of $\mathrm{End}_\Lambda(\mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda))$ such that for almost every[3] ν we have

$$\nu(\mathcal{E}(\mathcal{F})) = e(\nu(\mathcal{F}))\,. \tag{3.4}$$

The *space of* p*-stabilized* Λ*-adic cuspidal forms* is defined to be

$$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda) := \mathcal{E}\,\mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda)\,.$$

It follows from (3.4) that for a p-stabilized Λ-adic cuspidal form $\mathcal{F}$, the specialization $\nu(\mathcal{F})$ is a p-stabilized cuspidal form for almost all ν.

The next result will be crucial in §4.

Proposition 3.23. *[Wil88, Thm. 1.2.1] The space of* p*-stabilized* Λ*-adic cuspidal forms* $\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda)$ *is a free* Λ*-module of finite rank.*

Remark 3.24. *Hecke operators* for Λ-adic modular forms. For every integral ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, one can define a Λ-linear maps

$$\mathcal{T}(\mathfrak{a}) := \mathcal{T}_{\overline{\mathfrak{n}}}(\mathfrak{a}), \mathcal{S}(\mathfrak{a}) := \mathcal{S}_{\overline{\mathfrak{n}}}(\mathfrak{a})\colon\ \mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda) \to \mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\,|\,\Lambda)$$

[3] Throughout this note "for almost every ν"="for all but finitely many $\nu \in \mathfrak{X}$".

with the key property that for almost every ν we have

$$\nu(\mathcal{T}(\mathfrak{a})(\mathcal{F})) = T_{\mathfrak{n}p^r}(\mathfrak{a})(\nu(\mathcal{F})) \qquad \text{and} \qquad \nu(\mathcal{S}(\mathfrak{a})(\mathcal{F})) = S_{\mathfrak{n}p^r}(\mathfrak{a})(\nu(\mathcal{F}))\,,$$

if $\nu \in \mathfrak{X}_{r,k}$. The last formula relates the central character of $\nu(\mathcal{F})$ with $\boldsymbol{\psi}$ compatibly with the relation of Remark 3.21. It is precisely the desire of a formula of this kind what explains the choice in Definition 3.19 for the central character of $\nu(\mathcal{F})$.

Definition 3.25. Let $\mathfrak{n}_0$ be the greatest divisor of $\mathfrak{n}$ which is coprime to p.

i) We say that $\mathcal{F} \in \mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \Lambda)$ is a *Hecke eigenform* if, for all integral ideal $\mathfrak{a} \subseteq \mathcal{O}_F$, we have $\mathcal{T}(\mathfrak{a})(\mathcal{F}) = \lambda(\mathfrak{a}, \mathcal{F})(X)\,\mathcal{F}$ for some $\lambda(\mathfrak{a}, \mathcal{F})(X) \in \Lambda$.

ii) A Hecke eigenform $\mathcal{F} \in \mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \Lambda)$ is called *normalized* if $c(\mathcal{O}_F, \mathcal{F})(X) = 1$.

iii) A normalized Hecke eigenform $\mathcal{F} \in \mathcal{S}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \Lambda)$ is called a *newform* of level $\overline{\mathfrak{n}}$ if for almost every ν (equiv. for infinitely many ν) we have that $\nu(\mathcal{F})$ is a newform of level divisible by $\mathfrak{n}_0$.

For a normalized Hecke eigenform $\lambda(\mathfrak{p}, \mathcal{F})(X) = c(\mathfrak{p}, \mathcal{F})(X)$ for every prime ideal $\mathfrak{p} \subseteq \mathcal{O}_F$.

Remark 3.26. From now on (and specially in the next section), we will need to extend coefficients to $\mathcal{K}$. To this aim, set

$$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \mathcal{K}) := \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \Lambda) \otimes_\Lambda \mathcal{K}\,.$$

Then, it can be shown that the finite extension $\mathcal{K}$ of $\mathbb{Q}_p((X))$ can be chosen large enough so that

$$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \mathcal{K}) = \mathcal{K}\{\mathcal{F}(\mathfrak{a}z) \mid \mathcal{F} \text{ is a newform of level } \overline{\mathfrak{m}} \text{ with } \overline{\mathfrak{m}}\mathfrak{a} | \overline{\mathfrak{n}}\}\,.$$

The next result is crucial for our purposes. It is due to Hida for $k \geq 2$.

Theorem 3.27. *[Wil88, Thm. 3] Let $k \geq 1$, $r \geq 0$, and ζ a root of unity of order p^r. Let $\mathfrak{n} \subseteq \mathcal{O}_F$ be an integral ideal and let ϱ_r be as defined in (3.3). For every p-stabilized newform $f \in S_k^{\mathrm{ord}}(\mathfrak{n}, \psi\varrho_r\omega^{2-k} \,|\, \mathcal{O}[\zeta])$, where $\mathcal{O}$ is a finite extension of $\mathbb{Z}_p$ containing $\mathbb{Z}_p[\psi]$, there exist a finite extension Λ of $\mathbb{Z}_p[[X]]$, $\nu \in \mathfrak{X}_{k,r}$ (as in Remark 3.18), and a newform $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \Lambda)$ such that $\nu(\mathcal{F}) = f$.*

This talk is not oriented towards the proof of the above theorem. Instead, we will focus on the next result, which is due to Hida for $F = \mathbb{Q}$.

Theorem 3.28. *[Wil88, Thm. 4] Let $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \Lambda)$ be a p-stabilized newform. Then there is a unique continuous irreducible representation*

$$\varrho_{\mathcal{F}} \colon G_F \to \mathrm{GL}_2(\mathcal{K})$$

unramified outside $\mathfrak{n}p$ such that, for every prime $\mathfrak{q} \nmid \mathfrak{n}p$, one has

$$\mathrm{Tr}(\varrho_{\mathcal{F}})(\mathrm{Frob}_{\mathfrak{q}}) = c(\mathfrak{q}, \mathcal{F})(X)\,,$$
$$\det(\varrho_{\mathcal{F}})(\mathrm{Frob}_{\mathfrak{q}}) = \boldsymbol{\psi}(\mathfrak{q})N(\mathfrak{q})\,.$$

The proof of the above theorem, for which we will use all the theory developed so far, will be postponed until §4. We will conclude the section by showing how Theorem 3.28, together with Theorem 3.27, immediately implies the main Theorem 2.2.

Proof of Theorem 2.2 (case $F = \mathbb{Q}$). We start with two remarks:

- If $\varrho_{\mathbf{f},\lambda}$ exists as a representation into $\mathrm{GL}_2(K)$, with K a finite extension of $K_{\mathbf{f},\lambda}$, then Schur's Lemma guarantees that there is an equivalent representation with image in $\mathrm{GL}_2(\mathcal{O}_{\mathbf{f},\lambda})$.

- If $\varrho_{\mathbf{f},\lambda}$ exists, then it is irreducible (Ribet).

By Theorem 3.27, given a p-stabilized newform $\mathbf{f} \in S_k^{\mathrm{ord}}(\mathfrak{n}, \psi \,|\, \mathcal{O})$, there exists $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi}\omega^{k-2} \,|\, \Lambda)$ such that $\nu(\mathcal{F}) = \mathbf{f}$ for some $\nu \in \mathfrak{X}_{k,1}$. By Theorem 3.28, $\mathcal{F}$ has attached a continuous irreducible representation $\varrho_{\mathcal{F}}$. Consider the representation

$$\varrho_{\mathbf{f}} \colon G_F \xrightarrow{\varrho_{\mathcal{F}}} \mathrm{GL}_2(L) \xrightarrow{\nu} \mathrm{GL}_2(\overline{\mathbb{Q}}_p)\,.$$

It satisfies that

$$\begin{aligned} \mathrm{Tr}(\varrho_{\mathbf{f}})(\mathrm{Frob}_{\mathfrak{q}}) &= \nu(c(\mathfrak{q}, \mathcal{F})(X)) = c(\mathfrak{q}, \mathbf{f})\,, \\ \det(\varrho_{\mathbf{f}})(\mathrm{Frob}_{\mathfrak{q}}) &= \nu(\boldsymbol{\psi}(\mathfrak{q})N(\mathfrak{q})) = \psi(\mathfrak{q})N(\mathfrak{q})^{k-2}N(\mathfrak{q})\,, \end{aligned}$$

where we have used Remark 3.21. □

4 Sketch of the proof

4.1 Warm up: the proof for $F = \mathbb{Q}$

We set $F = \mathbb{Q}$ in this section. Then, the statement of Theorem 2.2 is contained in the results that we have seen in the first (weight $k \geq 2$; see [Del68]) and the second talks (weight $k = 1$; [DS74]). However, we consider remarkable the fact that Wiles' method recovers these results from the classical theory of Eichler-Shimura (weight $k = 2$), and we wish to describe this in detail in this short section.

In the present setting, an element $\mathbf{f} \in S_k(\mathfrak{n}, \psi)$ consists of a single function $f \colon \mathbb{H} \to \mathbb{C}$. If $\mathfrak{n} = (N)$ for $N \geq 1$, then $\psi \colon (\mathbb{Z}/N\mathbb{Z})^* \to \overline{\mathbb{Q}}^*$ is a Dirichlet character. The space $S_k(\mathfrak{n}, \psi)$ is what is usually denoted by

$$S_k(N, \psi) := S_k(\Gamma_0(N), \psi)\,.$$

Let $f \in S_k^{\mathrm{ord}}(N, \psi \,|\, \mathcal{O})$ and $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\overline{N}, \boldsymbol{\psi} \,|\, \Lambda)$, where the notation for the spaces is analogous to that used in (3.2). For an ideal (n) of $\mathbb{Z}$, for $n \geq 0$, let us use the notation $c_n(f)$ (resp. $c_n(\mathcal{F})(X)$) for the Fourier coefficient $c((n), f)$ (resp. $c((n), \mathcal{F})(X)$). Let $\mathcal{K}$ and Λ be as defined in Remark 3.18.

Proof of Theorem 3.28. Let $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\overline{N}, \boldsymbol{\psi} \,|\, \Lambda)$ be a p-stabilized newform. This means that there exist infinitely many $n \geq 1$ such that

$$f_n := \nu(\mathcal{F}) \in S_2^{\mathrm{ord}}(Np^n, \psi\varrho_n \,|\, \mathcal{O}[\zeta_n])$$

is a normalized Hecke eigenform for some $\nu \in \mathfrak{X}_{2,n}$. Let $P_{2,n}$ be the prime of Λ corresponding to $\nu \in \mathfrak{X}_{2,n}$. By the theory of Eichler-Shimura seen in the first talk, attached to f_n there is a continuous irreducible odd representation

$$\varrho_n \colon G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}[\zeta_n])$$

unramified outside Np satisfying that, for every $q \nmid Np$, one has

$$\begin{aligned} \mathrm{Tr}(\varrho_n)(\mathrm{Frob}_q) &= c_q(f) \equiv c_q(\mathcal{F})(X) \pmod{P_{2,n}}, \\ \det(\varrho_n)(\mathrm{Frob}_q) &= \psi(q)q^1 \equiv \boldsymbol{\psi}(q)q \pmod{P_{2,n}}. \end{aligned}$$

The statement now follows immediately from Theorem 3.11. □

4.2 The general case

Remark 4.1. If $d := [F \colon \mathbb{Q}]$ is odd, then Theorem 3.28 is proven proceeding as we did in §4.1. Let $\mathcal{F}$ be a p-stabilized Λ-adic newform. One observes that infinitely many specializations of $\mathcal{F}$ (for example, $\nu(\mathcal{F})$ with $\nu \in \mathfrak{X}_{k,1}$ for all but finitely many $k \geq 2$) have attached a representation by part $i)$ of Carayol's Theorem with the desired properties. Then, one just applies Theorem 3.11. We will therefore assume from now on that d is even.

Let us give a few words on the general strategy. Let $\mathcal{F} \in \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}, \boldsymbol{\psi} \,|\, \Lambda)$ be a p-stabilized Λ-adic newform. Choose a prime $\mathfrak{l}$, so that hypothesis $ii)$ of Theorem 2.3 with respect to $\mathfrak{l}$ and $\mathfrak{l}\overline{\mathfrak{n}}$ is satisfied. Consider a basis of newforms (with respect to $\mathfrak{l}$) of the space of forms of level $\overline{\mathfrak{n}}\mathfrak{l}$. By our choice of the prime $\mathfrak{l}$, Carayol's Theorem says that there are λ-adic representations attached to almost all specializations of each element in this basis. One obtains a Λ-adic representation attached to each element in this basis by patching the Carayol representations together using Theorem 3.11. By assembling the Λ-adic representations attached to each element of the basis, one obtains a representation ϱ on $(\mathbb{T} \otimes \mathcal{K}) \oplus (\mathbb{T} \otimes \mathcal{K})$, where $\mathbb{T}$ denotes the Hecke algebra. One then defines an ideal $I_{\mathcal{F}}$ of $\mathbb{T}$, such that for every prime ideal $I_{\mathcal{F}} \subseteq P \subseteq \mathbb{T}$ the representation ϱ modulo P is essentially the representation we are looking for reduced modulo a certain prime Q of Λ. Let ϱ_Q denote this representation. By varying $\mathfrak{l}$, one shows that infinitely many distinct such primes Q exist. One then concludes by patching together the corresponding representations ϱ_Q by using Theorem 3.11 again. The Hecke algebra $\mathbb{T}$ plays a fundamental role in relating our original $\mathcal{F}$ of level $\overline{\mathfrak{n}}$ with the basis of newforms (with respect to $\mathfrak{l}$) of the space of forms of level $\overline{\mathfrak{n}}\mathfrak{l}$.

Let $\mathfrak{l} \subseteq \mathcal{O}_F$ be a prime ideal such that $(\mathfrak{l}, \mathfrak{n}p) = 1$. As always, $\boldsymbol{\psi}$ comes from a ray class character ψ of modulus $\mathfrak{n}\infty$. Let Λ and $\mathcal{K}$ be as in Remark 3.26.

Definition 4.2. The space of p-stabilized Λ-adic *oldforms with respect to* $\mathfrak{l}$ is

$$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{old}} := \{\mathcal{G}_1(z)+\mathcal{G}_2(\mathfrak{l}z)\,|\,\mathcal{G}_1,\mathcal{G}_2\in\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}},\boldsymbol{\psi}\,|\,\mathcal{K})\}\,.$$

The space of p-stabilized Λ-adic *newforms with respect to* $\mathfrak{l}$ is

$$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}} := \mathcal{K}\left\{\mathcal{G}_i(\mathfrak{a}_{ij}z)\,\middle|\, \begin{array}{l}\mathcal{G}_i\in\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{m}}_i,\boldsymbol{\psi}\,|\,\Lambda)\text{ newform}\\ \text{and } \mathfrak{l}|\overline{\mathfrak{m}}_i,\ \mathfrak{a}_{ij}\overline{\mathfrak{m}}_i|\overline{\mathfrak{n}}\mathfrak{l}\end{array}\right\}. \tag{4.1}$$

We say that $\{\mathcal{G}_i(\mathfrak{a}_{ij}z)\}_{i,j}$ is a *special basis* for $\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}}$. Note that this special basis has a finite number of elements thanks to Proposition 3.23. By Remark 3.15, we may moreover assume that $(\mathfrak{a}_{ij},p)=1$.

One can show that there exists a decomposition

$$\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K}) = \mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{old}}\oplus\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}}\,, \tag{4.2}$$

which does not necessarily hold if we take coefficients in Λ instead of $\mathcal{K}$.

Definition 4.3. Set

$$\begin{aligned} H(\mathcal{F},\mathfrak{l}\,|\,\mathcal{K}) \;\; := \;\; &\{\mathcal{H}\in\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}}\,|\,\mathcal{H}=\mathcal{G}-u\,\mathcal{F}-v\,\mathcal{F}(\mathfrak{l}z),\\ &\text{with } \mathcal{G}\in\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\Lambda),\ u,v\in\mathcal{K}\}\,. \end{aligned}$$

The *congruence module for* $\mathcal{F}$ is

$$C(\mathcal{F},\mathfrak{l}\,|\,\mathcal{K}) := H(\mathcal{F},\mathfrak{l}\,|\,\mathcal{K})/(\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}}\cap\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\Lambda)),$$

and it measures how far the direct sum decomposition (4.2) fails to be a direct sum over Λ (see §4.3 for more information on the congruence module).

Proof of Theorem 3.28. Let $\mathbb{T}$ denote the ring generated over Λ by the Hecke operators $\mathcal{T}(\mathfrak{m})$, for $\mathfrak{m}$ prime to $\mathfrak{l}$, in $\mathrm{End}_{\mathcal{K}}(\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}})$. Set

$$I_{\mathcal{F}} = \mathrm{Ann}(C(\mathcal{F},\mathfrak{l}\,|\,\mathcal{K}))\subseteq\mathbb{T}\,.$$

Note that since

$$\mathcal{T}(\mathfrak{m})-c(\mathfrak{m},\mathcal{F})(X)\in I_{\mathcal{F}} \tag{4.3}$$

for each ideal $\mathfrak{m}$ prime to $\mathfrak{l}$, we have that $\mathbb{T}\,/I_{\mathcal{F}}\simeq\Lambda/\mathfrak{b}_{\mathcal{F},\mathfrak{l}}$ for some ideal $\mathfrak{b}_{\mathcal{F},\mathfrak{l}}\subseteq\Lambda$.

Let $\{\mathcal{G}_i(\mathfrak{a}_{ij}z)\}_{i,j}$ be a special basis for $\mathcal{S}^{\mathrm{ord}}(\overline{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}}$. By our choice[4] of $\mathfrak{l}$, and as in Remark 4.1 or in §4.1, Carayol's Theorem and Theorem 3.11 imply that there exists a Λ-adic representation $\varrho_{\mathcal{G}_i(\mathfrak{a}_{ij})}\colon G_F\to\mathrm{GL}_2(\mathcal{K})$ attached to $\mathcal{G}_i(\mathfrak{a}_{ij}z)$.

Endow the finite dimensional $\mathcal{K}$-vector space

$$A := \prod_{i,j}\mathcal{K}\,,$$

[4] We remark that $\mathfrak{l}$ is not in the support of the modulus of the central character of any specialization of $\mathcal{G}_i(\mathfrak{a}_{ij}z)$ (since $\mathfrak{l}$ was taken coprime to $\mathfrak{n}$) and that it divides $\mathfrak{a}_{ij}\overline{\mathfrak{m}}_i$ exactly (by the definition (4.1) of the $\mathfrak{l}$-newspace). Thus hypothesis *ii*) of Theorem 2.3 with respect to $\mathfrak{l}$ and $\mathfrak{a}_{ij}\overline{\mathfrak{m}}_i$ is satisfied for almost every specialization of $\mathcal{G}_i(\mathfrak{a}_{ij}z)$.

where the product runs over the elements defining the special basis, with an action of $\mathbb{T}$ by transport of structure. One can show that the map

$$\mathbb{T}\otimes\mathcal{K}\to A \qquad \text{induced by} \qquad \mathcal{T}\mapsto\prod_{i,j}c(\mathcal{O}_F,\mathcal{T}(\mathcal{G}_i(\mathfrak{a}_{ij}z))(X) \tag{4.4}$$

is an isomorphism of $\mathbb{T}\otimes\mathcal{K}$-modules. Let G_F act on

$$W:=A\oplus A$$

by means of $\bigoplus\varrho_{\mathcal{G}_i(\mathfrak{a}_{ij}z)}\otimes\mathcal{K}$. We obtain an odd representation

$$\varrho\colon G_F\to \mathrm{GL}_2(\mathbb{T}\otimes\mathcal{K})$$

such that, for any $\mathfrak{q}\nmid\mathfrak{n}\mathfrak{l}p$, one has

$$\begin{aligned}\mathrm{Tr}(\varrho)(\mathrm{Frob}_{\mathfrak{q}}) &= \textstyle\prod_{i,j}c(\mathfrak{q},\mathcal{G}_i(\mathfrak{a}_{ij}z))(X)=\\ &= \textstyle\prod_{i,j}c(\mathcal{O}_F,\mathcal{T}(\mathfrak{q})(\mathcal{G}_i(\mathfrak{a}_{ij}z)))(X)\\ &= \mathcal{T}(\mathfrak{q})\in\mathbb{T}\,,\end{aligned}$$

where we have used the isomorphism $\mathbb{T}\otimes\mathcal{K}\simeq A$ given by (4.4) for the last equality. Its associated odd pseudo-representation π has thus values in $\mathbb{T}$. By reduction modulo $I_{\mathcal{F}}$, we get an odd pseudo-representation $\overline{\pi}$ with values in $\mathbb{T}/I_{\mathcal{F}}\simeq\Lambda/\mathfrak{b}_{\mathcal{F},\mathfrak{l}}$. Because of (4.3), for any $\mathfrak{q}\nmid\mathfrak{n}\mathfrak{l}p$, we have

$$\mathrm{Tr}(\overline{\pi})(\mathrm{Frob}_{\mathfrak{q}})=\mathcal{T}(\mathfrak{q})\equiv c(\mathfrak{q},\mathcal{F})(X)\in\Lambda/\mathfrak{b}_{\mathcal{F},\mathfrak{l}}\,.$$

Choose a prime ideal $\mathfrak{b}_{\mathcal{F},\mathfrak{l}}\subseteq Q\subseteq\Lambda$. Let π_Q denote the pseudo-representation $\overline{\pi}$ reduced modulo Q. By Lemma 3.3, associated to π_Q there exists an odd representation

$$\varrho_Q\colon G_F\to\mathrm{GL}_2(\mathcal{O}_Q)\,,$$

where $\mathcal{O}_Q$ denotes the integral closure of Λ/Q in its field of fractions, such that for any $\mathfrak{q}\nmid\mathfrak{n}\mathfrak{l}p$, we have

$$\mathrm{Tr}(\varrho_Q)(\mathrm{Frob}_{\mathfrak{q}})=c(\mathfrak{q},\mathcal{F})(X)\in\Lambda/Q\,.$$

The proof continues with a technical argument to ensure that, by making distinct choices of $\mathfrak{l}$, we may find *infinitely many distinct* primes $\mathfrak{b}_{\mathcal{F},\mathfrak{l}}\subseteq Q\subseteq\Lambda$. One then concludes by patching all the representations ϱ_Q together by means of Theorem 3.11. We give some of the ideas used to show the existence of this infinite set of primes Q in §4.3. $\square$

4.3 On the existence of infinitely many primes Q

Keep the notations and assumptions (on Λ and $\mathcal{K}$, and on $\mathfrak{l}$, $\mathfrak{p}$, $\mathfrak{n}$) of the previous section. The idea is to gain control on the size of $\mathfrak{b}_{\mathcal{F},\mathfrak{l}}$, so that the existence of infinitely many primes $\mathfrak{b}_{\mathcal{F},\mathfrak{l}}\subseteq Q\subseteq\Lambda$ can be guaranteed. To make precise what we mean by "control on the size" let us introduce some notation. For a fractional ideal $\mathfrak{a}$ of $\mathcal{K}$, define

$$\operatorname{div}(\mathfrak{a}) := \sum v_{P_i}(\mathfrak{a}) P_i$$

where the sum is taken over prime ideals P_i of height one and v_{P_i} denotes the discrete valuation at P_i. In the proposition below, we will show that there exists a fractional ideal $\mathfrak{a}_{\mathcal{F},\mathfrak{l}}$ of $\mathcal{K}$ such that $C(\mathcal{F},\mathfrak{l}\,|\,\mathcal{K}) \simeq \mathfrak{a}_{\mathcal{F},\mathfrak{l}}/\Lambda$. It then follows from the definition of $\mathfrak{b}_{\mathcal{F},\mathfrak{l}}$ that

$$-\operatorname{div}(\mathfrak{b}_{\mathcal{F},\mathfrak{l}}) \leq \operatorname{div}(\mathfrak{a}_{\mathcal{F},\mathfrak{l}})\,.$$

Thus, control on $\mathfrak{a}_{\mathcal{F},\mathfrak{l}}$ will provide control on $\mathfrak{b}_{\mathcal{F},\mathfrak{l}}$. This is achieved in the next proposition (see also the conjecture below).

Proposition 4.4. *[Wil88, Thm. 1.6.1] Let*

$$\begin{aligned} w_{\mathfrak{l}} := w_{\mathfrak{l}}(X) \quad &:= \quad (\alpha_{\mathfrak{l}}^2 - \boldsymbol{\psi}(\mathfrak{l}))(\beta_{\mathfrak{l}}^2 - \boldsymbol{\psi}(\mathfrak{l})) = \\ &= \quad -\boldsymbol{\psi}(\mathfrak{l})(c(\mathfrak{l},\mathcal{F})(X)^2 - \boldsymbol{\psi}(\mathfrak{l})(1+N(\mathfrak{l}))^2) \in \Lambda\,, \end{aligned}$$

where $\alpha_{\mathfrak{l}} := \alpha_{\mathfrak{l}}(X)$ and $\beta_{\mathfrak{l}} := \beta_{\mathfrak{l}}(X)$ are the roots of $x^2 - c(\mathfrak{l},\mathcal{F})(X)x + \boldsymbol{\psi}(\mathfrak{l})N(\mathfrak{l})$. There exists a fractional ideal $\mathfrak{a}_{\mathcal{F},\mathfrak{l}}$ of $\mathcal{K}$ such that $C(\mathcal{F},\mathfrak{l}\,|\,\mathcal{L}) \simeq \mathfrak{a}_{\mathcal{F},\mathfrak{l}}/\Lambda$ and

$$\operatorname{div}(w_{\mathfrak{l}}^{-1}) \leq \operatorname{div}(\mathfrak{a}_{\mathcal{F},\mathfrak{l}}) \leq \operatorname{div}(w_{\mathfrak{l}}^{-1}) + \operatorname{div}(V) + c\operatorname{div}(1+N(\mathfrak{l}))\,, \tag{4.5}$$

where $V \in \Lambda$ and $c \in \mathbb{Z}$ are both independent of $\mathfrak{l}$.

Conjecture 4.5. *[Wil88, p. 555] For P of Λ not above p, we have $v_P(w_{\mathfrak{l}}^{-1}) = v_P(\mathfrak{a}_{\mathcal{F},\mathfrak{l}})$.*

Ideas on the proof of Proposition 4.4. Let us prove the left inequality of (4.5). This amounts to showing that $w_{\mathfrak{l}}$ annihilates the image of the injective map

$$\gamma\colon C(\mathcal{F},\mathfrak{l}\,|\,\mathcal{K}) \to \mathcal{K}/\Lambda\,, \qquad \gamma(\mathcal{H}) = u\,,$$

where $\mathcal{H} = \mathcal{G} - u\,\mathcal{F} - v\,\mathcal{F}(\mathfrak{l}z)$ is as in Definition 4.3. By Lemma 4.6 below, we have that $\mathcal{S}^{\mathrm{ord}}(\bar{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}}$ is the kernel of the operator $\mathcal{T}(\mathfrak{l})^2 - \boldsymbol{\psi}(\mathfrak{l})$. Define the operator $\mathcal{U}(\mathfrak{l}) := (\alpha_{\mathfrak{l}}^2 - \mathcal{T}(\mathfrak{l})^2)(\beta_{\mathfrak{l}}^2 - \mathcal{T}(\mathfrak{l})^2)$ and consider the equalities

$$w_{\mathfrak{l}}\mathcal{H} = \mathcal{U}(\mathfrak{l})(\mathcal{H}) = \mathcal{U}(\mathfrak{l})(\mathcal{G}) \in \mathcal{S}^{\mathrm{ord}}(\bar{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\Lambda)\,. \tag{4.6}$$

The first equality is due to the fact that $\mathcal{H}$ is an element of $\mathcal{S}^{\mathrm{ord}}(\bar{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\mathcal{K})^{\mathrm{new}}$, and the second equality follows from the fact that $\mathcal{U}(\mathfrak{l})(\mathcal{F}) = \mathcal{U}(\mathfrak{l})(\mathcal{F}(\mathfrak{l}z)) = 0$. In particular, we have that $\omega_{\mathfrak{l}}\mathcal{H} \in \mathcal{S}^{\mathrm{ord}}(\bar{\mathfrak{n}}\mathfrak{l},\boldsymbol{\psi}\,|\,\Lambda)$. Taking $c(\mathcal{O}_F,\cdot)(X)$ coefficients to the equality

$$w_{\mathfrak{l}}\mathcal{G} - w_{\mathfrak{l}}u\,\mathcal{F} - w_{\mathfrak{l}}v\,\mathcal{F}(\mathfrak{l}z) = w_{\mathfrak{l}}\mathcal{H}$$

we obtain

$$\begin{aligned} -w_{\mathfrak{l}}u \quad &= \quad -w_{\mathfrak{l}}u\cdot c(\mathcal{O}_F,\mathcal{F})(X) - w_{\mathfrak{l}}v\cdot c(\mathcal{O}_F,\mathcal{F}(\mathfrak{l}z))(X) \equiv \\ &\equiv \quad c(\mathcal{O}_F, w_{\mathfrak{l}}\mathcal{H})(X) \equiv 0 \pmod{\Lambda}\,. \end{aligned}$$

The other inequality requires a lot of deep and hard work. □

Lemma 4.6. *[Wil88, Lem. 1.4.5] Let $k \geq 2$, let $\mathfrak{q}$ be a prime not dividing $\mathfrak{n}$, let ψ be defined modulo $\mathfrak{n}$, and let $\mathbf{f} \in S_k^{\mathrm{ord}}(\mathfrak{n}\mathfrak{q}, \psi)$ be a normalized Hecke eigenform. Then $\mathbf{f}$ is new with respect to $\mathfrak{l}$ if and only if $c(\mathfrak{q}, \mathbf{f})^2 - \psi(\mathfrak{q})N(\mathfrak{q})^{k-2} = 0$. In this case, $\mathbf{f}$ is locally (unramified) special at $\mathfrak{q}$.*

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