Sato–Tate groups of abelian threefolds

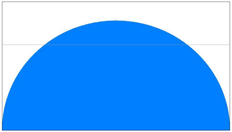
Francesc Fité (IAS), Kiran S. Kedlaya (UCSD), A.V. Sutherland (MIT)

Arithmetic of low-dimensional abelian varieties.
ICERM, 5th June 2019.

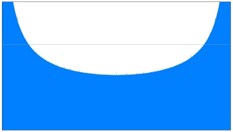
Sato–Tate groups of elliptic curves

- k a number field.
- E/k an elliptic curve.
- The Sato–Tate group $\mathrm{ST}(E)$ is defined as:
  - $\mathrm{SU}(2)$ if E does not have CM.
  - $\mathrm{U}(1) = \left\{ \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix} : u \in \mathbb{C}, |u| = 1 \right\}$ if E has CM by $M \subseteq k$.
  - $N_{\mathrm{SU}(2)}(\mathrm{U}(1))$ if E has CM by $M \not\subseteq k$.
- Note that $\mathrm{Tr}\colon \mathrm{ST}(E) \to [-2, 2]$. Denote $\mu = \mathrm{Tr}_*(\mu_{\mathrm{Haar}})$.

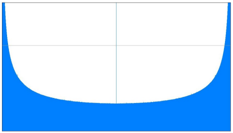
SU(2)



U(1)



$N(\mathrm{U}(1))$



The Sato–Tate conjecture for elliptic curves

- Let $\mathfrak{p}$ be a prime of good reduction for E. The normalized Frobenius trace satisfies

$$\overline{a}_{\mathfrak{p}} = \frac{N(\mathfrak{p}) + 1 - \#E(\mathbb{F}_{\mathfrak{p}})}{\sqrt{N(\mathfrak{p})}} = \frac{\mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}} \,|\, V_\ell(E))}{\sqrt{N(\mathfrak{p})}} \in [-2, 2] \qquad (\text{for } \mathfrak{p} \nmid \ell)$$

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The sequence $\{\overline{a}_{\mathfrak{p}}\}_{\mathfrak{p}}$ is equidistributed on $[-2, 2]$ w.r.t μ.

- If $\mathrm{ST}(E) = \mathrm{U}(1)$ or $N(\mathrm{U}(1))$: Known in full generality (Hecke, Deuring).
- Known if $\mathrm{ST}(E) = \mathrm{SU}(2)$ and k is totally real. (Barnet-Lamb, Geraghty, Harris, Shepherd-Barron, Taylor);
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Toward the Sato–Tate group: the ℓ-adic image

- Let A/k be an abelian variety of dimension $g \geq 1$.
- Consider the ℓ-adic representation attached to A

$$\varrho_{A,\ell}\colon G_k \to \mathrm{Aut}_{\psi_\ell}(V_\ell(A)) \simeq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)\,.$$

- Serre defines $\mathrm{ST}(A)$ in terms of $\mathcal{G}_\ell = \varrho_{A,\ell}(G_k)^{\mathrm{Zar}}$.
- For $g \leq 3$, Banaszak and Kedlaya describe $\mathrm{ST}(A)$ *in terms of endomorphisms*.
- By Faltings, there is a G_k-equivariant isomorphism

$$\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell \simeq \mathrm{End}_{\mathcal{G}_\ell^0}(\mathbb{Q}_\ell^{2g})\,.$$

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- From now on, assume $g \leq 3$.

Definition

$\mathrm{ST}(A) \subseteq \mathrm{USp}(2g)$ is a maximal compact subgroup of $\mathrm{TL}(A) \otimes_{\mathbb{Q}} \mathbb{C}$.

- Note that

$$\mathrm{ST}(A)/\,\mathrm{ST}(A)^0 \simeq \mathrm{TL}(A)/\,\mathrm{TL}(A)^0 \simeq \mathrm{Gal}(F/k)\,.$$

where F/k is the minimal extension such that $\mathrm{End}(A_F) \simeq \mathrm{End}(A_{\overline{\mathbb{Q}}})$. We call F the endomorphism field of A.

- To each prime $\mathfrak{p}$ of good reduction for A, one can attach a conjugacy class $x_{\mathfrak{p}} \in X = \mathrm{Conj}(\mathrm{ST}(A))$ s.t. $\mathrm{Char}(x_{\mathfrak{p}}) = \mathrm{Char}\left(\frac{\varrho_{A,\ell}(\mathrm{Frob}_{\mathfrak{p}})}{\sqrt{N\mathfrak{p}}}\right)$.

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Sato–Tate axioms for $g \leq 3$

The Sato–Tate axioms for a closed subgroup $G \subseteq \mathrm{USp}(2g)$ for $g \leq 3$ are:

Hodge condition (ST1)

There is a homomorphism $\theta\colon \mathrm{U}(1) \to G^0$ such that $\theta(u)$ has eigenvalues u and $\overline{u}$ each with multiplicity g. The image of such a θ is called a *Hodge circle*. Moreover, the Hodge circles generate a dense subgroup of G^0.

Rationality condition (ST2)

For every connected component $H \subseteq G$ and for every irreducible character $\chi\colon \mathrm{GL}_{2g}(\mathbb{C}) \to \mathbb{C}$:

$$\int_H \chi(h)\mu_{\mathrm{Haar}} \in \mathbb{Z},$$

where μ_{Haar} is normalized so that $\mu_{\mathrm{Haar}}(G^0) = 1$.

Lefschetz condition (ST3)

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Proposition

If $G = \mathrm{ST}(A)$ for some A/k with $g \leq 3$, then G satisfies the ST axioms.

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- Axioms (ST1), (ST2) are expected for general g. But not (ST3)!

Remark

- Up to conjugacy, 3 subgroups of USp(2) satisfy the ST axioms.
- All 3 occur as ST groups of elliptic curves defined over number fields.
- Only 2 of them occur as ST groups of elliptic curves defined over totally real fields.

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If $G = \mathrm{ST}(A)$ for some A/k with $g \leq 3$, then G satisfies the ST axioms.

$$\begin{array}{rcl} \text{Mumford–Tate conjecture} & \rightsquigarrow & \text{(ST1)} \\ \text{“Rationality” of } \mathcal{G}_\ell & \rightsquigarrow & \text{(ST2)} \\ \text{Bicommutant property of } \mathcal{G}_\ell^0 & \rightsquigarrow & \text{(ST3)} \end{array}$$

- Axioms (ST1), (ST2) are expected for general g. But not (ST3)!

Remark

- Up to conjugacy, 3 subgroups of USp(2) satisfy the ST axioms.
- All 3 occur as ST groups of elliptic curves defined over number fields.
- Only 2 of them occur as ST groups of elliptic curves defined over totally real fields.

Sato–Tate groups for $g = 2$

Theorem (F.-Kedlaya-Rotger-Sutherland; 2012)

- Up to conjugacy, 55 subgroups of USp(4) satisfy the ST axioms.
- 52 of them occur as ST groups of abelian surfaces over number fields.
- 35 of them occur as ST groups of abelian surfaces over totally real number fields.
- 34 of them occur as ST groups of abelian surfaces over $\mathbb{Q}$.
- Above can replace "abelian surfaces" with "Jacobians of genus 2 curves".

Corollary

The degree of the endomorphism field of an abelian surface over a number field divides 48.

Theorem (F.-Guitart; 2016)

There exists a number field (of degree 64) over which all 52 ST groups can be realized.

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Sato–Tate conjecture for $g = 2$

Theorem (Johansson, N. Taylor; 2014-19)

For $g = 2$ and k totally real, the ST conjecture holds for 33 of the 35 possible ST groups.

- The missing cases are $\mathrm{USp}(4)$ and $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$.
- The case $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$ corresponds to an abelian surface A/k, which is either:
  - $\mathrm{Res}^L_k(E)$, where L/k quadratic and E/L an e.c. which is not a k-curve; or
  - absolutely simple with real multiplication not defined over k.
- If $k = \mathbb{Q}$, the ST conjecture holds for $N(\mathrm{SU}(2) \times \mathrm{SU}(2))$ (N. Taylor).

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Sato–Tate groups for $g = 3$

Theorem(F.-Kedlaya-Sutherland; 2019)

- Up to conjugacy, 433 subgroups of USp(6) satisfy the ST axioms.
- Only 410 of them occur as Sato–Tate groups of abelian threefolds over number fields.

Corollary

The degree of the endomorphism field $[F : \mathbb{Q}]$ of an abelian threefold over a number field divides 192, 336, or 432.

- This refines a previous result of Guralnick and Kedlaya, which asserts

$$[F : \mathbb{Q}] \mid 2^6 \cdot 3^3 \cdot 7 = \mathrm{Lcm}(192, 336, 432)\,.$$

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Classification: identity components

(ST1) and (ST3) allow 14 possibilities for $G^0 \subseteq \mathrm{USp}(6)$:

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Notations:

- For $d \in \{2,3\}$ and $H \in \{\mathrm{SU}(2), \mathrm{U}(1)\}$:

$$H_d = \{\mathrm{diag}(u, \overset{d}{\dots}, u) \,|\, u \in H\}$$

- For $d \in \{1,3\}$:

$$\mathrm{U}(d) = \begin{pmatrix} \mathrm{U}(d)^{\mathrm{St}} & 0 \\ 0 & \overline{\mathrm{U}(d)}^{\mathrm{St}} \end{pmatrix} \subseteq \mathrm{USp}(2d)$$

- Note in particular that

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Classification: From Lie groups to finite groups

General strategy to recover the possibilities for G from G^0:

- Compute $N = N_{\mathrm{USp}(6)}(G^0)$ and ${}^{N}/_{G^0}$.
- Use

$$\left\{ \begin{array}{c} \mathcal{G} \subseteq \mathrm{USp}(6) \text{ with } \mathcal{G}^0 = G^0 \\ \text{satisfying (ST2)} \end{array} \right\} /\sim \;\longleftrightarrow\; \left\{ \begin{array}{c} \text{finite } H \subseteq N/G^0 \text{ s.t.} \\ HG^0 \text{ satisfies (ST2)} \end{array} \right\} /\sim$$

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  - Split case: G^0 can be written as in (*) and

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Classification: cases depending on G^0

$$\text{Genuine dim. 3 cases} \begin{cases} \mathrm{USp}(6) \\ \mathrm{U}(3) \end{cases}$$

$$\text{Split cases} \begin{cases} \mathrm{SU}(2) \times \mathrm{USp}(4) \\ \mathrm{U}(1) \times \mathrm{USp}(4) \\ \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \\ \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1) \\ \mathrm{SU}(2) \times \mathrm{SU}(2)_2 \\ \mathrm{SU}(2) \times \mathrm{U}(1)_2 \\ \mathrm{U}(1) \times \mathrm{SU}(2)_2 \\ \mathrm{U}(1) \times \mathrm{U}(1)_2 \end{cases}$$

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Classification: From G^0 to G

- Genuine cases: USp(6), U(3), N(U(3)).
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  The set on the right can be recovered from the ST group classifications in dimensions 1 and 2. This accounts for 211 groups.

- Non-split cases:

G^0	N/G^0	$\#\mathcal{A}$
SU(2) × SU(2) × SU(2)	S_3	4
U(1) × U(1) × U(1)	$(C_2 \times C_2 \times C_2) \rtimes S_3$	33
$\mathrm{SU}(2)_3$	SO(3)	11
$\mathrm{U}(1)_3$	$\mathrm{PSU}(3) \rtimes C_2$	171

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$\mathrm{SU}(2)_3$	$\mathrm{SO}(3)$	11
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Classification: From G^0 to G

- Genuine cases: USp(6), U(3), $N(\mathrm{U}(3))$.
- Split cases: Since $N/G^0 \simeq N_1/G^{0,1} \times N_2/G^{0,2}$ we have

$$\mathcal{A} = \left\{ \begin{array}{c} H \subseteq N/G^0 \text{ finite s.t.} \\ HG^0 \text{ satisfies (ST2)} \end{array} \right\} /\sim \;\longleftrightarrow\; \left\{ \begin{array}{c} H = H_1 \times_{H'} H_2 \text{ with} \\ H_i \subseteq N_i/G^{0,i} \text{ finite s.t.} \\ H_i G^{0,i} \text{ satisfies (ST2)} \end{array} \right\} /\sim$$

  The set on the right can be recovered from the ST group classifications in dimensions 1 and 2. This accounts for 211 groups.
- Non-split cases:

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- The above injection is seen to be a bijection a posteriori.
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Classification: Invariants

- Only 210 distinct pairs $(G^0, G/G^0)$.
- Define the (i,j,k)-**th moment**, for $i,j,k \geq 0$, as

$$\mathrm{M}_{i,j,k}(G) := \dim_{\mathbb{C}} \left((\wedge^1\mathbb{C}^6)^{\otimes i} \otimes (\wedge^2\mathbb{C}^6)^{\otimes j} \otimes (\wedge^3\mathbb{C}^6)^{\otimes k} \right)^G \in \mathbb{Z}_{\geq 0}\,.$$

- The tuple $\{\mathrm{M}_{i,j,k}(G)\}_{i+j+k\leq 6}$ attains 432 values. It only conflates a pair of groups G_1, G_2, for which however

$$G_1/G_1^0 \simeq \langle 54, 5\rangle \not\simeq \langle 54, 8\rangle \simeq G_2/G_2^0\,.$$

- In fact, $\mathrm{M}_{i,j,k}(G_1) = \mathrm{M}_{i,j,k}(G_2)$ for all i,j,k !
- In total, the 433 groups have 10988 connected components (4 for $g=1$ and 414 for $g=2$).
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$$\mathrm{M}_{i,j,k}(G) := \dim_{\mathbb{C}} \left((\wedge^1\mathbb{C}^6)^{\otimes i} \otimes (\wedge^2\mathbb{C}^6)^{\otimes j} \otimes (\wedge^3\mathbb{C}^6)^{\otimes k} \right)^G \in \mathbb{Z}_{\geq 0}\,.$$

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Realization

- By Shimura, if A/k has CM by M, then $F = M^* k$. This rules out:
  - 20 groups in the case $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$.
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- This leaves 410 groups.
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Realization of the maximal groups

- Genuine cases (2 max. groups):
  - USp(6): generic case. Eg.: $y^2 = x^7 - x + 1/\mathbb{Q}$.
  - $N(\mathrm{U}(3))$: Picard curves. Eg.: $y^3 = x^4 + x + 1/\mathbb{Q}$.
- Split cases (13 max. groups):
  Maximality ensures the triviality of the fiber product, i.e.

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  where G_1 and G_2 are realizable in dimensions 1 and 2.
- Non-split cases (18 max. groups):
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  - All such G satisfy
    $$G/G^0 \hookrightarrow \mathrm{GL}_3(\mathcal{O}_M) \rtimes \mathrm{Gal}(M/\mathbb{Q})$$
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  - Reinterpret
    $$G/G^0 \hookrightarrow \mathrm{Aut}(E_M^3) \rtimes \mathrm{Gal}(M/\mathbb{Q})$$
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  - This gives a 1-cocycle
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  - There exists $L/\mathbb{Q}$ such that $G/G^0 \simeq \mathrm{Gal}(L/\mathbb{Q})$.
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Open questions

- Realizability over totally real fields?
- Realizability over $\mathbb{Q}$?
- Existence of a number field over which all 410 groups can be realized?
- Realizability via principally polarized abelian thereefolds?
- Realizability via Jacobians of genus 3 curves?
  - Partial answer: At least 22 of the 33 maximal groups can be realized via Jacobians...

G/G^0	$\#(G/G^0)$	C with $\mathrm{ST}(\mathrm{Jac}(C))$
$(C_4 \times C_4) \rtimes S_3 \times C_2$	192	Twist of the Fermat quartic
$\mathrm{PSL}(2,7) \times C_2$	336	Twist of the Klein quartic
$(C_6 \times C_6) \rtimes S_3 \times C_2$	432	?
$E_{216} \times C_2$	432	?