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4th BTech Math Summer Camp

ARITHMETIC

1 Easy problems

Problem 1. *Let $a, b, c \in \mathbb{Z}$. Show that if a divides b and c, then a divides $b+c$.*

Problem 2. *Show that if n, $n+2$ and $n+4$ are primes, then $n = 3$.*

Problem 3. *Find all natural solutions $(x, y) \in \mathbb{N} \times \mathbb{N}$ of the system of equations*

$$\begin{cases} xy = 51840 \\ \gcd(x, y) = 24 \end{cases}$$

Problem 4. *Show that there is a unique natural number $n \in \mathbb{N}$ with only 2 distinct prime divisors, 6 distinct divisors, and such that the sum of these 6 divisors is 28.*

Problem 5. *Establish Bézout's identity for the numbers $a = 7658$ and $b = 3853$.*

Problem 6. *Find all integral solutions $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ of*

- $24x + 106y = 3$.
- $24x + 106y = 2$.

Problem 7. *Show that if p is prime, then p divides $\binom{p}{k}$ for $1 \leq k \leq p-1$.*

Problem 8. *What is the exponent of 2 in the factorization into prime numbers of $29!$?*

Problem 9. *For $n \in \mathbb{N}$, show that if $2^n - 1$ is prime, then n is prime.*

Problem 10. *We say that $n \in \mathbb{N}$ is a* perfect number *if it is the sum of its divisors strictly less than n itself. Show that if $2^n - 1$ is prime, then $2^{n-1}(2^n - 1)$ is a perfect number.*

Problem 11. *Prove that a natural number n is:*

- *divisible by 3 if and only if the sum of the digits of n in the decimal expression is divisible by 3.*
- *divisible by 11 if and only if the sum of the digits in the odd positions minus the sum of the digits in the even positions in the decimal expression of n is divisible by 11.*

Hint: Use congruences.

Problem 12. *Compute $2^{17} \pmod{25}$.*

Hint: Use that $17 = 2^4 + 2^0$.

Problem 13. *Show that for natural numbers $n > 1$, a and b, we have*

$$\gcd(n^a - 1, n^b - 1) = n^{\gcd(a,b)} - 1\,.$$

2 Intermediate problems

Problem 14. *Assign a natural number to each of the faces of a cube. Assign to each of the vertices of the cube, the product of the numbers assigned to each of the 3 faces meeting at that vertex. If the sum of the numbers assigned to the vertices of the cube is 1001, which is the sum of the numbers assigned to the faces of the cube?*

Problem 15. *Show that:*

i) There exist infinitely many primes of the form $4n - 1$.

ii) There exist infinitely many primes of the form $6n - 1$.

Problem 16. *Find all integral solutions* $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ *of the equation*

$$p(x + y) = xy\,,$$

where p *is a prime number.*

Problem 17. *Find all integral solutions* $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ *of the equation*

$$x^3 + y^3 = 1729.$$

Problem 18. *Show that for any prime* p*, we have*

$$a^p \equiv b^p \pmod{p} \quad \Rightarrow \quad a^p \equiv b^p \pmod{p^2}\,.$$

Hint: Use Problem 7.

Problem 19. *Show that there is no nonconstant polynomial* $f(x) \in \mathbb{Z}[x]$ *such that* $f(a)$ *is a prime for every* $a \in \mathbb{Z}$.

Problem 20. *Show that* $2222^{5555} + 5555^{2222}$ *is divisible by 7.*

Problem 21. *Consider the sequence defined by* $a_1 = 7$ *and* $a_{n+1} = 7^{a_n}$ *for* $n \geq 1$. *Which is the last digit of the number* a_{1001}*?*

Problem 22. *Let* $p > 5$ *be a prime. Show that* $p - 4$ *is not the fourth power of an integer.*

Problem 23. *Consider the function* $f : \mathbb{N} \to \mathbb{N}$ *such that*

- $f(1) = 1$.
- $f(n) = f(n/2)$ *if* n *is even.*
- $f(n) = f((n - 1)/2) + 1$ *if* n *is odd.*

Compute the maximum value of $f(n)$ *for* $n \leq 2018$.

Problem 24. *Show that* $3^n + 2 \cdot 17^n$ *is not a perfect square for any* $n \in \mathbb{N}$.

3 Hard problems

Problem 25. *Let $d_1 < d_2 \cdots < d_k$ be the positive divisors of a natural number $n > 1$. Show that*

$$d_1d_2 + d_2d_3 + \cdots + d_{k-1}d_k < n^2 .$$

Problem 26. *Find all natural solutions $(n, m) \in \mathbb{N} \times \mathbb{N}$ of*

$$(m + 1)^n - 1 = m!$$

Problem 27. *Determine the last 3 digits of $2003^{2002^{2001}}$.*

Problem 28. *Show that the function $f : \mathbb{N} \to \mathbb{N}$ defined by*

$$f(n) = \left\lfloor n + \sqrt{n} + \frac{1}{2} \right\rfloor$$

takes all the values of $\mathbb{N}$ except of the perfect squares.

Problem 29. *Show that the sequence*

$$1, 11, 111, 1111, \ldots$$

contains an infinite subsequence all of whose terms are pairwise relatively prime.

Problem 30. *Show that $n^5 + n^4 + 1$ is not a prime for any $n > 1$.*

Problem 31. *Show that for every $n \in \mathbb{N}$, we have*

$$\binom{3n}{0} + \binom{3n}{3} + \cdots + \binom{3n}{3n-3} + \binom{3n}{3n} = \frac{1}{3}(2(-1)^n + 2^{3n}) .$$

Problem 32. *We say that $(x, y, z) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is a Pythagorean triple if $x^2 + y^2 = z^2$. Show that (x, y, z) is a Pythagorean triple if and only if there exist natural numbers m, n, k with $m > n$ and $\gcd(n, m) = 1$ such that*

$$x = k \cdot (m^2 - n^2), \qquad y = k \cdot (2mn), \qquad z = k \cdot (m^2 + n^2).$$

Problem 33. *Show that the equation $x^4 + y^4 = z^4$ has no solutions in positive integers x, y, z.*

4 Solutions

Solution of 1: If a divides b and c, there exist b' and c' in $\mathbb{Z}$ such that $b = b'a$ and $c = c'a$. Then $b + c = a(b' + c')$.
Solution of 2: Let r be the remainder of n divided by 3. If $r = 0$, then $n = 3$ since n is prime. If $r = 1$, then $n+2 > 3$ would be divisible by 3. If $r = 2$, then $n + 4 > 3$ would be divisible by 3.
Solution of 3: Write $x = 24t$ and $y = 24u$, where t and u are relatively prime. Then $24^2 tu = 51840$ and therefore $tu = 90$. It only remains to find all coprime factorizations of 90 into two factors:

- $90 = 45 \cdot 2$. Then $x = 1080$ and $y = 48$.
- $90 = 9 \cdot 10$. Then $x = 216$ and $y = 240$.
- $90 = 18 \cdot 5$ Then $x = 432$ and $y = 120$.

Permuting x and y we obtain another triple of solutions.
Solution of 4: We are told that n is of the form $p^a q^b$, for prime numbers a and b. Since $6 = (a+1)(b+1)$, we may assume that $a = 1$ and $b = 2$. The condition

$$2 \cdot 14 = 4 \cdot 7 = 28 = 1 + q + q^2 + p + pq + pq^2 = (1 + p)(1 + q + q^2)$$

impies that $7 = 1 + q + q^2$ (otherwise $p = 1$ or $p = 6$ would not be prime). Thus $q = 2$ and $p = 3$, whereby $n = 12$.
Solution of 5: Bézout's identity reads

$$ax + by = 1\,,$$

where $x = -883$, $y = 1755$.
Solution of 6: The first equation has no solution, since $\gcd(24, 106) = 2 \nmid 3$. The general solution of the second equation is

$$\begin{cases} x = -22 + 53k \\ y = 5 - 12k \end{cases}$$

for $k \in \mathbb{Z}$.
Solution of 7: Note that

$$\binom{p}{k} = \frac{1 \cdot 2 \cdot 3 \cdot \dots \cdot (p-1) \cdot p}{(1 \cdot 2 \cdot \dots \cdot k)(1 \cdot 2 \cdot \dots \cdot (p-k))}$$

and that p divides the numerator but not the denominator of the above expression.
Solution of 8: The exponent of 2 is:

$$\left\lfloor \frac{29}{2} \right\rfloor + \left\lfloor \frac{29}{2^2} \right\rfloor + \left\lfloor \frac{29}{2^3} \right\rfloor + \left\lfloor \frac{29}{2^4} \right\rfloor = 14 + 7 + 3 + 1 = 25.$$

Solution of 9: Suppose that $n = ab$ with $a, b > 1$. The factorization

$$x^b - 1 = (x - 1)(x^{b-1} + x^{b-2} + \cdots + x + 1)$$

for $x = 2^a$ gives the factorization

$$2^n - 1 = (2^a - 1)(2^{a(b-1)} + x^{a(b-2)} + \cdots + 2^a + 1)\,.$$

Solution of 10: If $2^n - 1$ is prime, then the proper divisors of $2^{n-1}(2^n - 1)$ are

$$1, 2, \ldots, 2^{n-1}, 2^n - 1, 2(2^n - 1), \ldots, 2^{n-2}(2^n - 1).$$

The sum of these divisors is

$$2^n - 1 + (2^n - 1)(2^{n-1} - 1) = (2^n - 1)2^{n-1}\,.$$

Solution of 11: Let $a_t a_{t-1} \ldots a_1 a_0$ be the decimal expression of n. Then:

$$\begin{aligned} n &= a_t 10^t + a_{t-1} 10^{t-1} + \cdots + a_1 10 + a_0 \equiv \\ &\equiv a_t + a_{t-1} + \cdots + a_1 + a_0 \pmod 3 \end{aligned}$$

$$\begin{aligned} n &= a_t 10^t + a_{t-1} 10^{t-1} + \cdots + a_1 10 + a_0 \equiv \\ &\equiv a_t (-1)^t + a_{t-1}(-1)^{t-1} + \cdots + a_1(-1) + a_0 \pmod{11} \end{aligned}$$

Solution of 12: We use the modular exponentiation method. Note that:
$2^2 \equiv 4 \pmod{25}$
$2^4 \equiv (2^2)^2 \equiv 16 \pmod{25}$
$2^8 \equiv (2^4)^2 \equiv 16^2 \equiv 9^2 = 81 \equiv 6 \pmod{25}$
$2^{16} \equiv (2^8)^2 \equiv 6^2 \equiv 36 \equiv 11 \pmod{25}$
Thus:

$$2^{17} \equiv 2^{16} \cdot 2 \equiv 11 \cdot 2 \equiv 22 \pmod{25}.$$

Solution of 13: Without loss of generality we may assume that $a > b$. Note that

$$\gcd(n^a - 1, n^b - 1) = \gcd(n^a - 1 - n^{a-b}(n^b - 1), n^b - 1) = \gcd(n^{a-b} - 1, n^b - 1)$$

Iterating this calculation, we obtain the result by Euclid's algorithm.
Solution of 14: Let a and f be the numbers assigned to the top and bottom faces of the cube, respectively. Let b, c, d, e be the numbers assigned to the lateral faces of the cube. Then:

$$7{\cdot}11{\cdot}13 = 1001 = abc+abe+acd+ade+bcf+bef+cdf+def = (a+f)(b+d)(c+e)$$

Thus $\{7, 11, 13\} = \{a+f, b+d, c+e\}$ and $a+b+c+d+e+f = 7+11+13 = 31$.
Solution of 15: First note that a prime is either of the form $4n - 1$ or $4n + 1$. Suppose that there was only a finite number of primes of the form $4n - 1$, say $p_1, \ldots, p_r$. Consider the product

$$4p_1 \ldots p_r - 1$$

Since it is of the form $4n - 1$ and larger than the p_i, it cannot be a prime. It is not possible that all of its prime factors are of the form $4n+1$, since the product of two numbers of the form $4n + 1$ is again of the form $4n + 1$. Thus there is a prime of the form $4n-1$, that is a certain p_i, dividing $4p_1 \dots p_r - 1$. But then p_i divides 1, which is a contradiction.

The oder case is analogous once one notes that every prime is either of the form $6n+1$ or $6n-1$.

Solution of 16: One solution is $(x, y) = (0, 0)$. Suppose that $x, y \neq 0$. Suppose that p divides x and write $x = pt$, for a nonzero $t \in \mathbb{Z}$. We have

$$p(pt + y) = pty, \qquad \text{that is} \qquad pt = y(t-1).$$

Since $\gcd(t, t-1) = 1$, either $p|y$ or $p = t-1$ and $y = t$.

If $p|y$, then $y = pu$ for a nonzero $u \in \mathbb{Z}$ and then $t = u(t-1)$. Since $t \neq 0$, so is $t - 1 \neq 0$. Then

$$u = \frac{t}{t-1} = 1 + \frac{1}{t-1}.$$

Thus $t-1 = 1$ and then $x = 2p$ and $y = 2p$. If $p = t-1$ and $y = t$ we obtain $y = p+1$ and $x = py = p(p+1)$.

Since the equation is symmetric in x and y, we also have the solution $x = p+1$ and $y = p(p+1)$.

Solution of 17: Note that

$$7 \cdot 13 \cdot 19 = 1729 = x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

Consider the system:

$$\begin{cases} x + y = a \\ x^2 - xy + y^2 = b \\ a \cdot b = 7 \cdot 13 \cdot 19 \end{cases}$$

One easily sees that $x^2 - xy + y^2 > 0$ and thus $b > 0$ an $a > 0$. Combining the first two equations, we find

$$3x^2 - 3ax + a^2 - b = 0.$$

Therefore

$$x = \frac{3a \pm \sqrt{12b - 3a^2}}{6}.$$

One easily checks that the only possible integral values of $a > 0$ and $b > 0$ for which $a \cdot b = 7 \cdot 13 \cdot 19$ and $12b - 3a^2$ is a perfect square are $a = 13$ and $b = 7 \cdot 19$; $a = 19$ and $b = 7 \cdot 13$. We obtain that $\{x, y\} = \{9, 10\}$ or $\{x, y\} = \{1, 12\}$.

Solution of 18: Just note that

$$a^p - b^p = a^p - (a + b - a)^p = p(b-a)a^{p-1} + \cdots + p(b-a)^{p-1}a + (b-a)^p.$$

Thus if $p|a^p - b^p$, then $p|(b-a)$. But then $p^2|a^p - b^p$.

Solution of 19: Suppose that such a polynomial exists. Take $a_0 \in \mathbb{Z}$. Then $f(a_0) = p_0$ is a prime. For every $k \in \mathbb{Z}$, we have that $f(a_0 + kp_0)$ is prime and also

$$f(a_0 + kp_0) \equiv f(a_0) \equiv 0 \pmod{p_0}.$$

It follows that $f(a_0 + kp_0) = p_0$ for every k. Thus the polynomial $g(x) = f(a_0 + xp_0) - p_0$ has a zero at every integer. Since a nonzero polynomial can have only a finite number of zeros, we find that $g(x)$ is identically 0, which implies that $f(x)$ is constant. This is a contradiction.

Solution of 20: By Fermat's little theorem, for $a \in \mathbb{Z}$ such that $\gcd(a,7) = 1$, we have that $a^6 \equiv 1 \pmod 7$. Then

$$2222^{5555} + 5555^{2222} \equiv 3^{5555} + 4^{2222} \equiv 3^5 + 4^2 \equiv 5 + 2 \equiv 0 \pmod 7 .$$

Solution of 21: By Euler's theorem, we need to compute $a_{1001} \equiv 7^b \pmod{10}$, where $b \equiv a_{1000} \pmod 4$ (recall that $\varphi(10) = 4$). But

$$b \equiv a_{1000} \equiv 7^{a_{999}} \equiv (-1)^{a_{999}} \equiv -1 \equiv 3 \pmod 4 ,$$

where we have used that all the a_i are obviously odd. Thus

$$a_{1001} \equiv 7^3 \equiv 3 \pmod{10} .$$

Solution of 22: Suppose that $p - 4 = q^4$ for some $q \in \mathbb{Z}$. Then

$$p = q^4 + 4q^2 + 4 - 4q^2 = (q^2+2)^2 - (2q)^2 = (q^2+2q+2)(q^2-2q+2)$$

We next show that the two factors in the above expression are > 1. It is enough to show it for the second. But in this case we have:

$$q^2 - 2q + 2 = (q-1)^2 + 1 > 1$$

since if $q = 1$, then $p = 5$, contrary to the hypothesis.
Solution of 23: $f(n)$ is the sum of the digits of n expressed in base 2. The maximum value of $f(n)$ for $n \leq 2018$ is thus $f(1023) = 10$.
Solution of 24: Note that $3^4 \equiv 1 \pmod{16}$. Then

$$3^n + 2 \cdot 17^n \equiv \begin{cases} 3 + 2 \equiv 5 \pmod{16} & \text{if } n \equiv 1 \pmod 4 \\ 9 + 2 \equiv 11 \pmod{16} & \text{if } n \equiv 2 \pmod 4 \\ 11 + 2 \equiv 13 \pmod{16} & \text{if } n \equiv 3 \pmod 4 \\ 1 + 2 \equiv 3 \pmod{16} & \text{if } n \equiv 0 \pmod 4 \end{cases}$$

But note that the squares modulo 16 are $0, 1, 4, 9$.
Solution of 25: It is enough to show that

$$\frac{1}{d_1 d_2} + \frac{1}{d_2 d_3} + \cdots + \frac{1}{d_{k-1} d_k} < 1 .$$

But note that

$$\frac{1}{d_1 d_2} + \frac{1}{d_2 d_3} + \cdots + \frac{1}{d_{k-1} d_k} \leq \frac{d_2 - d_1}{d_1 d_2} + \frac{d_3 - d_2}{d_2 d_3} + \cdots + \frac{d_k - d_{k-1}}{d_{k-1} d_k} =$$

$$\frac{1}{d_1} - \frac{1}{d_2} + \frac{1}{d_2} - \frac{1}{d_3} + \cdots + \frac{1}{d_{k-1}} - \frac{1}{d_k} = 1 - \frac{1}{n} < 1 .$$

Solution of 26: We first try a few values. For $m = 1$, we get $n = 1$. For $m = 2$, we get $n = 1$. For $m = 3$, there is no solution. For $m = 4$, we get $n = 2$.

Suppose that $m > 4$. We will see that there are no solutions in this case. Note that $m + 1$ must be prime and, in particular, m must be even. Then

$$m! = (m+1)^n - 1 = (m+1-1)((m+1)^{n-1} + (m+1)^{n-2} + \cdots + (m+1) + 1).$$

Therefore
$$(m-1)! \equiv n \pmod m$$
Since m is even and $m \geq 6$, we have that $m|(m-1)!$. Hence $m|n$ and in particular $m \leq n$. But then
$$(m+1)^n \geq (m+1)^m > m^m + 1 > m! + 1\,.$$
Solution of 27: Since $\varphi(1000) = \varphi(5^3)\varphi(2^3) = 5^2(5-1)2^2(2-1) = 16{\cdot}25 = 400$, we need to determine
$$2002^{2001} \equiv 2^{2001} \equiv 16 \cdot 2^{1997} \pmod{16 \cdot 25}\,.$$
To find this value, we need to determine $2^{1997} \pmod{25}$. Since $\varphi(25) = 20$, we have
$$2^{1997} \equiv 2^{17} \equiv 22 \pmod{25}\,.$$
The last congruence above has been proven in Problem 12. We thus get:
$$2002^{2001} \equiv 2^{2001} \equiv 16 \cdot 22 \equiv 352 \pmod{16 \cdot 25}\,.$$
Therefore
$$2003^{2002^{2001}} \equiv 3^{352} \equiv 9^{176} \equiv (10-1)^{176} \equiv \binom{176}{2}10^2 - \binom{176}{1}10 + 1 \equiv$$
$$\equiv 0 - 760 + 1 \equiv 241 \pmod{1000}\,.$$
Solution of 28: Suppose that f misses the value $m \in \mathbb{N}$. This means that there exists $n \in \mathbb{N}$ such that
$$n + \sqrt{n} + \frac{1}{2} < m \qquad m + 1 < n + 1 + \sqrt{n+1} + \frac{1}{2}$$
This implies that
$$\sqrt{n} < m - n - \frac{1}{2} < \sqrt{n+1}.$$
That is
$$n < (m-n)^2 - (m-n) + \frac{1}{4} < n+1 \qquad \text{or} \qquad n - \frac{1}{4} < (m-n)^2 - (m-n) < n + 1 - \frac{1}{4}.$$
But since m and n are natural numbers, this means that $(m-n)^2 - (m-n) = n$, that is, $m = (m-n)^2$.

To see that f takes all the values which are not perfect squares, fix $k \in \mathbb{N}$. Note that the set
$$\{n \in \mathbb{N} \,|\, f(n) \leq k^2 + k\}$$
has cardinality k^2. This means that up to $k^2 + k$, the function f misses k values. But up to $k^2 + k$, there are precisely k perfect squares. Thus f misses precisely the perfect squares.

Solution of 29: Denote by x_n that the nth term of the sequence of the statement of the problem. That is, set
$$x_n = \frac{10^n - 1}{9}\,.$$

Observe that if $\gcd(n, m) = 1$, then by Problem 13 we have that

$$\gcd(x_n, x_m) = \frac{1}{9}\gcd(10^n - 1, 10^m - 1) = \frac{9}{9} = 1\,.$$

It is thus enough to find an infinite sequence of natural numbers which are pairwise relatively prime. Take for example the sequence of prime numbers.

Solution of 30: The primitive cubic roots of unity $\omega, \overline{\omega}$ are the roots of the polynomial

$$X^2 + X + 1$$

They satisfy $\omega^3 = (\overline{\omega})^3 = 1$. Thus:

$$\omega^5 + \omega^4 + 1 = \omega^2 + \omega + 1 = 0$$

and similarly for $\overline{\omega}$. We deduce that the polynomial $X^2 + X + 1$ must divide $X^5 + X^4 + 1$. By performing the polynomial division we obtain

$$X^5 + X^4 + 1 = (X^2 + X + 1)(X^3 - X + 1).$$

Solution of 31: Consider the sums

$$\begin{aligned} A &= \tbinom{3n}{0} + \tbinom{3n}{3} + \dots \\ B &= \tbinom{3n}{1} + \tbinom{3n}{4} + \dots \\ C &= \tbinom{3n}{2} + \tbinom{3n}{5} + \dots \end{aligned}$$

Let 1, ω, and $\overline{\omega}$ denote the three cubic roots of unity. We easily see that

$$\begin{aligned} (1+1)^{3n} &= A + B + C\,, \\ (1+\omega)^{3n} &= A + B\omega + C\overline{\omega}\,, \\ (1+\overline{\omega})^{3n} &= A + B\overline{\omega} + C\omega\,. \end{aligned}$$

Since $1 + \omega + \overline{\omega} = 0$, by adding the three equations, we obtain that

$$2^{3n} + (-\overline{\omega})^{3n} + (-\omega)^{3n} = 3A \quad \text{or} \quad 2^{3n} + 2(-1)^n = 3A\,.$$

Solution of 32: Note that (x, y, z) is a Pythagorean triple if and only if the point $(X, Y) = (x/z, y/z)$ of rational coordinates lies on the intersection of the circle C of equation

$$X^2 + Y^2 = 1$$

and the first quadrant of the plane.

Note that the points (X, Y) with rational coordinates lying on the intersection of the circle C and the first quadrant of the plane are in bijection with the lines ℓ of rational slope $0 < t < 1$ passing through the point $(-1, 0)$.

To determine this set of points, note that ℓ is given by the equation $Y = t(X + 1)$. Thus, its intersection point other than $(-1, 0)$ with C satisfies

$$X^2 + t^2(X + 1)^2 = 1$$

$$(X + 1)(X - 1 + t^2(X + 1)) = 0$$

Therefore $X - 1 + t^2(X+1) = 0$, that is, $X(1+t^2) = 1 - t^2$. Substituting in $Y = t(X+1)$, we obtain

$$(X, Y) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right) \qquad \text{with } t \in \mathbb{Q}\,.$$

But we can write $t = n/m$ for $n, m \in \mathbb{Z}$ with $\gcd(n, m) = 1$. The condition $0 < t < 1$, says that $n, m \in \mathbb{N}$ and that $m > n$. Thus we have

$$\left(\frac{x}{z}, \frac{y}{z} \right) = \left(\frac{m^2 - n^2}{m^2 + n^2}, \frac{2mn}{m^2 + n^2} \right),$$

which completes the proof.
Solution of 33: We will show a stronger result, that is, $x^4 + y^4 = z^2$ has no solutions in positive integers. Assume that such a solution exists. Take the solution for which z has the smallest value. We must have that x, y, z are coprime, since otherwise by dividing we would obtain an even smaller solution. Since (x^2, y^2, z) is a Pythagorean triple, by Problem 32 we have that

$$x^2 = 2pq\,, \qquad y^2 = p^2 - q^2\,, \qquad z = p^2 + q^2\,,$$

for some coprime $p > q$. Note that the equation $y^2 = p^2 - q^2$ gives rise to another Pythagorean triple and hence

$$q = 2ab\,, \qquad y = a^2 - b^2\,, \qquad p = a^2 + b^2\,,$$

for some coprime $a > b$. Then

$$x^2 = 2pq = 4ab(a^2 + b^2)$$

If p is a prime such that $p|a$ or $p|b$, then it cannot divide $a^2 + b^2$, since a and b are coprime. Hence ab and $a^2 + b^2$ are coprime and thus they are perfect squares. Since a and b are coprime, so are a and b perfect squares. Therefore

$$P^2 = a^2 + b^2 = A^4 + B^4$$

But note that

$$P^2 = a^2 + b^2 = p < p^2 + q^2 = z$$

and thus $P < z$. This is a contradiction with the minimality of z.