# Local-global principles for quadratic and polyquadratic twists of abelian varieties

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# Local-global principles for isogenies of abelian varieties

$K$ a number field.

$A, B/K$ abelian varieties of dimension $g \geq 1$.

$\Sigma$ the set of primes of bad reduction of $A$ and $B$.

$\forall \mathfrak{p} \not\in \Sigma$, denote by $A_{\mathfrak{p}}, B_{\mathfrak{p}}/K(\mathfrak{p})$ the reductions of $A, B$ modulo $\mathfrak{p}$.

**Notation**

$\forall' \mathfrak{p} =$ For every prime ideal of $\mathcal{O}_K$ outside a 0 density set containing $\Sigma$.

**Faltings isogeny theorem**

$A$ and $B$ are isogenous if and only if $A_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$ are isogenous $\forall' \mathfrak{p}$.

**Theorem (Khare- Larsen; 2020)**

$\overline{A}$ and $\overline{B}$ are isogenous if and only if $\overline{A}_{\mathfrak{p}}$ and $\overline{B}_{\mathfrak{p}}$ are isogenous $\forall' \mathfrak{p}$.
Here $\overline{A} := A \times_K \overline{K}$, $\overline{A}_{\mathfrak{p}} := A_{\mathfrak{p}} \times_{K(\mathfrak{p})} \overline{K(\mathfrak{p})}$.

# Polyquadratic twists

$F = K$ or $K(\mathfrak{p})$.

$A, B/F$ abelian varieties.

Category of abelian varieties up to isogeny:

- Objects: abelian varieties.
- Morphisms: $\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.

We say that $B$ is a twist of $A$ if there exists an algebraic field extension $L/F$ and an isogeny

$$\varphi : B_L \to A_L\,.$$

Here $A_L := A \times_F L$, $B_L := B \times_F L$.

We say that $B$ is a polyquadratic twist of $A$ (of degree $2^r$) if it is a twist for which the extension $L/F$ can be taken as the compositum of $r$ quadratic extensions.

# Quadratic twists

Let $G_F := \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of $F$.

Weil descent

$$
\begin{array}{ccc}
(\varphi : B_L \to A_L) & \longmapsto & (\sigma \mapsto {}^\sigma\varphi \circ \varphi^{-1}) \\
\{\text{Twists of } A \text{ (up to } F\text{-isogeny)}\} & \xleftrightarrow{1:1} & H^1(G_F, \mathrm{Aut}^0(A_{\overline{F}})) \\
\big\uparrow & & \big\uparrow \\
\{\text{Quadratic twists of } A \text{ (up to } F\text{-isogeny)}\} & \xleftrightarrow{1:1} & H^1(G_F, \{\pm 1\})
\end{array}
$$

Note that $H^1(G_F, \{\pm 1\}) = \mathrm{Hom}(G_F, \{\pm 1\})$.

We denote by $A_\chi$ the twist of $A$ attached to $\chi \in \mathrm{Hom}(G_F, \{\pm 1\})$.

We say that $B$ is a quadratic twist of $A$ if $B$ is (isogenous to) $A_\chi$ for some $\chi$.

# Quadratic twists

Alternative more explicit description:

Write $L := \overline{F}^{\ker(\chi)}$

$$A_\chi = \begin{cases} \text{complement of } A \text{ in } \mathrm{Res}_{L/F}(A) & \text{if } \chi \text{ is nontrivial} \\ A & \text{if } \chi \text{ is trivial.} \end{cases}$$

Remark

Not every polyquadratic twist of degree 2 is a quadratic twist.

Example

$A^2$ and $A \times A_\chi$ are polyquadratic twists of degree 2, but in general they will not be quadratic twists.

# Main results

$K$ a number field and $A, B/K$ abelian varieties of dimension $g \geq 1$.

**Theorem 1 (F.; 2021)**

Suppose that $g \leq 3$.
$A, B$ are quadratic twists if and only if $A_\mathfrak{p}, B_\mathfrak{p}$ are quadratic twists $\forall' \mathfrak{p}$.

**Example (E. Costa)**

The above is false for $g = 4$: The Jacobians of

$$y^2 = x^9 + x/\mathbb{Q}\,, \qquad y^2 = x^9 + 16x/\mathbb{Q}$$

are locally quadratic twists at all odd primes, but they are not quadratic twists.

**Theorem 2 (F.-Perucca; 2022)**

Suppose that $g \leq 2$.
$A, B$ are polyquadratic twists if and only if $A_\mathfrak{p}, B_\mathfrak{p}$ are polyquadratic twists $\forall' \mathfrak{p}$.
Moreover the above is false for $g = 3$.

# Representation theoretic setting

$E$ a topological field.

$G$ a compact topological group.

$\varrho, \varrho' : G \to \mathrm{GL}_r(E)$ semisimple continuous representations.

We say that $\varrho$ and $\varrho'$ are quadratic twists if $\varrho' \simeq \chi \otimes \varrho$ holds for some $\chi \in \mathrm{Hom}(G, \{\pm 1\})$.

We say that $\varrho$ and $\varrho'$ are polyquadratic twists if

$$\varrho \simeq \bigoplus_{i=1}^{t} \varrho_i \qquad \text{and} \qquad \varrho' \simeq \bigoplus_{i=1}^{t} \varrho'_i \,,$$

where $\varrho_i, \varrho'_i : G \to \mathrm{GL}_{r_i}(E)$ are quadratic twists for all $i$.

**Proposition**

$\varrho, \varrho'$ are polyquadratic twists if and only if $\varrho|_H \simeq \varrho'|_H$ for some $H \trianglelefteq G$ such that $G/H$ is a finite abelian group of exponent dividing 2.

# Example

$K$ a number field.

$A, B/K$ abelian varieties of dimension $g \geq 1$.

$\ell$ a prime. The $\ell$-adic Tate module of $A$ is

$$T_\ell(A) := \varprojlim_r A[\ell^r](\overline{K}) \simeq \mathbb{Z}_\ell^{2g}\,, \qquad V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\,.$$

We denote by $\varrho_{A,\ell}$ the representation of $G_K$ afforded by $V_\ell(A)$.

Faltings isogeny thm. (as on slide 1) with the Brauer-Nesbitt thm. imply:

**Proposition**

$A, B$ are isogenous if and only if $\varrho_{A,\ell} \simeq \varrho_{B,\ell}$.

$A, B$ are quadratic twists if and only if $\varrho_{A,\ell}\,, \varrho_{B,\ell}$ are quadratic twists.

$A, B$ are polyquadratic twists if and only if $\varrho_{A,\ell}\,, \varrho_{B,\ell}$ are polyquadratic twists.

# Local versions

$E$ a topological field & $G$ a compact topological group.

$\varrho, \varrho' : G \to \mathrm{GL}_r(E)$ semisimple continuous representations.

We say that $\varrho$ and $\varrho'$ are locally quadratic twists if for every $s \in G$ there exists $\epsilon_s \in \{\pm 1\}$ such that

$$\det(1 - \varrho(s)T) = \det(1 - \epsilon_s \varrho'(s)T)\,.$$

We say that $\varrho$ and $\varrho'$ are locally polyquadratic twists if for every $s \in G$

$$\det(1 - \varrho(s^2)T) = \det(1 - \varrho'(s^2)T)\,.$$

Remark

$\varrho$ and $\varrho'$ (poly)quadratic twists $\Longrightarrow$ $\varrho$ and $\varrho'$ locally (poly)quadratic twists.

Proposition

$A_\mathfrak{p}, B_\mathfrak{p}$ (poly)quadratic twists $\forall' \mathfrak{p} \iff \varrho_{A,\ell}, \varrho_{B,\ell}$ locally (poly)quadratic twists.

# Local versions: Representation theoretic description

$\varrho, \varrho'$ are locally polyquadratic twists if and only if

$$\mathrm{Sym}^2 \varrho - \wedge^2 \varrho \simeq \mathrm{Sym}^2 \varrho' - \wedge^2 \varrho'$$

as virtual representations.

$\varrho, \varrho'$ are locally quadratic twists if and only if for all $n$

$$\varrho \otimes \varrho \simeq \varrho' \otimes \varrho' , \quad \wedge^{2n} \varrho \simeq \wedge^{2n} \varrho' , \quad \varrho \otimes \wedge^{2n+1} \varrho \simeq \varrho' \otimes \wedge^{2n+1} \varrho' . \quad (*)$$

Some of the relations in (*) are redundant. For $\deg(\varrho) = \deg(\varrho') = 4$, then $\varrho, \varrho'$ are locally quadratic twists if and only if

$$\mathrm{Sym}^2 \varrho \simeq \mathrm{Sym}^2 \varrho' , \quad \wedge^2 \varrho \simeq \wedge^2 \varrho' .$$

# Question

Are the below implications in fact equivalences?

1) $\varrho$ and $\varrho'$ quadratic twists $\Longrightarrow$ $\varrho$ and $\varrho'$ locally quadratic twists.

The converse implication is true if $\deg(\varrho) = 2$ (Ramakrishnan) or odd;
False for $\deg(\varrho) = 4$ (Chidambaran) or 6.

2) $\varrho$ and $\varrho'$ polyquadratic twists $\Longrightarrow$ $\varrho$ and $\varrho'$ locally polyquadratic twists.

With Perucca, we show that the converse implication is true if $\deg(\varrho) \leq 2$, but false for $\deg(\varrho) \geq 3$.

### Corollary

Suppose that $g = 1$.
$A$ and $B$ are quadratic twists if and only if $A_{\mathfrak{p}}, B_{\mathfrak{p}}$ are quadratic twists $\forall' \mathfrak{p}$.

### Remark

The $\varrho, \varrho'$ in the above counterexamples in degrees 4 and 6 do *not* correspond to $\ell$-adic representations of abelian surfaces or threefolds.

# Ramakrishnan's theorem

Goal: Sketch the proof of Theorem 1

Theorem (Ramakrishnan)

If $\varrho, \varrho' : G \to \mathrm{GL}_2(E)$ are locally quadratic twists, then they are quadratic twists.

Proof

The hypothesis implies $\mathrm{ad}^0(\varrho) \simeq \mathrm{ad}^0(\varrho') \quad \& \quad \det(\varrho) \simeq \det(\varrho')$

We may assume that $\varrho, \varrho'$ are irreducible (otherwise it is an easy exercise).

Suppose that $\mathrm{ad}^0_\varrho$ is reducible. In this case, there is $H \trianglelefteq G$ with $[G : H] = 2$ such that:

$$\varrho \simeq \mathrm{Ind}_H^G(\nu)\,, \varrho' \simeq \mathrm{Ind}_H^G(\nu')\,,$$

where $\nu, \nu'$ are characters of $H$ such that $\nu'/\nu$ extends to a character $\chi$ of $G$. Then

$$\chi \otimes \varrho \simeq \mathrm{Ind}_H^G\left(\frac{\nu'}{\nu} \otimes \nu\right) \simeq \varrho'\,.$$

Suppose that $\mathrm{ad}^0_\varrho$ is irreducible. One has:

$$\mathrm{ad}^0_\varrho \otimes \mathrm{ad}^0_{\varrho'} \oplus 1 \simeq \mathrm{Sym}^2(\varrho \otimes \varrho') \otimes \det(\varrho)^{-1} \otimes \det(\varrho')^{-1} \subseteq (\varrho \otimes \varrho') \otimes (\varrho \otimes \varrho')^\vee .$$

Since the multiplicity of 1 in the LHS is $> 1$, we have that $\varrho \otimes \varrho'$ is reducible.

We claim that $\varrho \otimes \varrho'$ contains a 1-dimensional constituent $\nu$.

If otherwise $\varrho \otimes \varrho' \simeq \tau \oplus \tau'$ with $\deg(\tau) = \deg(\tau') = 2$, then $\det(\tau)$ would be a 1-dimensional contituent of

$$\bigwedge^2(\varrho \otimes \varrho') \simeq \mathrm{Sym}^2(\varrho) \otimes \det(\varrho') \oplus \mathrm{Sym}^2(\varrho') \otimes \det(\varrho) .$$

Then

$$(\nu \det(\varrho)^{-1}) \otimes \varrho \simeq \nu \otimes \varrho^\vee \simeq \varrho' .$$

# Rajan's theorem

## Theorem (Rajan)

Let $\varrho, \varrho : G_K \to \mathrm{GL}_r(\mathbb{Q}_\ell)$ be semisimple. Suppose:

- $\varrho(G_K)^{\mathrm{Zar}}$ is connected.
- $\mathrm{Dens}\big(\{\mathfrak{p} \mid \mathrm{Tr}(\varrho(\mathrm{Frob}_\mathfrak{p})) = \mathrm{Tr}(\varrho'(\mathrm{Frob}_\mathfrak{p}))\}\big) > 0$.

Then there exists a finite $L/K$ such that $\varrho|_{G_L} \simeq \varrho'|_{G_L}$.

## Corollary

If $A$ and $B$ are locally quadratic twists, then $A$ and $B$ are twists. In particular, $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \simeq \mathrm{End}(B_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$.

Let $K_A$ denote the minimal extension such that $\mathrm{End}(A_{K_A}) = \mathrm{End}(A_{\overline{\mathbb{Q}}})$.

## Proposition

If $A$ and $B$ are locally quadratic twists, then $K_A = K_B$.

Indeed, if $A$ and $B$ are locally quadratic twists, then

$$(\mathrm{Tr}\,\varrho_A(\mathrm{Frob}_{\mathfrak{p}}))^2 = (\mathrm{Tr}\,\varrho_B(\mathrm{Frob}_{\mathfrak{p}}))^2 \qquad \forall' \mathfrak{p}\,.$$

By Chebotarev

$$\varrho_A \otimes \varrho_A \simeq \varrho_B \otimes \varrho_B$$

Then for any extension $L/K$

$$\mathrm{End}(A_L) \otimes \mathbb{Q}_\ell \simeq (\varrho_A \otimes \varrho_A^\vee)^{G_L} \simeq (\varrho_B \otimes \varrho_B^\vee)^{G_L} \simeq \mathrm{End}(B_L) \otimes \mathbb{Q}_\ell\,.$$

## Sato-Tate group philosophy

For $g \leq 3$, if $A$ and $B$ are locally quadratic twists and the Sato-Tate conjecture *were* true, then $\mathrm{ST}(A) = \mathrm{ST}(B)$. Since

$$\mathrm{ST}(A)/\,\mathrm{ST}(A)^0 \simeq \mathrm{Gal}(K_A/K)\,, \qquad \mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{R} \text{ is determined by } \mathrm{ST}^0(A)\,,$$

it was reasonable to expect that

$$\mathrm{Gal}(K_A/K) \simeq \mathrm{Gal}(K_B/K) \quad \text{and} \quad \mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{R} \simeq \mathrm{End}(B_{\overline{\mathbb{Q}}}) \otimes \mathbb{R}\,.$$

# The case $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \simeq \mathbb{Z}$

The proof of Theorem 1 is by cases on the possibilities for $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$.

## The case $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \simeq \mathbb{Z}$

By Rajan's Theorem, there is a finite extension $L/K$ such that

$$\mathbb{Q}_\ell \simeq \mathrm{Hom}(A_L, B_L) \otimes \mathbb{Q}_\ell \simeq \mathrm{Hom}_{G_L}(\varrho_{A,\ell}, \varrho_{B,\ell}) \simeq (\varrho_{A,\ell}^\vee \otimes \varrho_{B,\ell})^{G_L}\,.$$

$(\varrho_{A,\ell}^\vee \otimes \varrho_{B,\ell})^{G_L}$ affords a character $\chi$ of $\mathrm{Gal}(L/K)$, which in fact is quadratic.

It will suffice to see that

$$\mathrm{Hom}_{G_K}(\varrho_{B,\ell}, \chi \otimes \varrho_{A,\ell}) \neq 0\,.$$

Note that

$$\mathrm{Hom}_{G_K}(\varrho_{B,\ell}, \chi \otimes \varrho_{A,\ell}) \simeq \mathrm{Hom}_{G_K}(\varrho_{A,\ell}^\vee \otimes \varrho_{B,\ell}, \chi) \neq 0\,.$$

# The case $A_{\overline{\mathbb{Q}}} \sim E^2$

Suppose we are in the almost antagonic case:

$A_{\overline{\mathbb{Q}}} \sim E^2$, where $E/\overline{\mathbb{Q}}$ is an elliptic curve without CM.

Theorem (F.-Guitart)

There exists a finite Galois extension $L/K$, a number field $M$, and

- An Artin representation $\theta : \mathrm{Gal}(L/K) \to \mathrm{GL}_2(M)$.
- For every $\ell$ totally split in $M$, a strongly absolutely irreducible $M$-rational $\ell$-adic representation $\varrho : G_K \to \mathrm{GL}_2(\mathbb{Q}_\ell)$

such that $\varrho_{A,\ell} \simeq \theta \otimes_{\mathbb{Q}_\ell} \varrho$.

Using the previous theorem, we can write

$$\varrho_{A,\ell} \simeq \theta \otimes \varrho\,, \qquad \varrho_{B,\ell} \simeq \theta' \otimes \varrho'\,.$$

After enlarging $L/K$, we can assume that

$$\theta, \theta' : \mathrm{Gal}(L/K) \to \mathrm{GL}_2(M) \quad \text{and} \quad \varrho_{A,\ell}|_{G_L} \simeq \varrho_{B,\ell}|_{G_L}\,.$$

In particular $\varrho|_{G_L} \simeq \varrho'|_{G_L}$. Hence there exists a character $\chi$ of $\mathrm{Gal}(L/K)$ such that $\varrho' \simeq \chi \otimes \varrho$.

That $A$ and $B$ are locally quadratic twists means that

$$\det(1 - \theta' \otimes \chi \otimes \varrho(\mathrm{Frob}_\mathfrak{p})T) = \det(1 \pm \theta \otimes \varrho(\mathrm{Frob}_\mathfrak{p})T) \qquad \forall' \mathfrak{p}\,.$$

Let $\alpha_{1,\mathfrak{p}}, \alpha_{2,\mathfrak{p}}$ be the eigenvalues of $\varrho(\mathrm{Frob}_\mathfrak{p})$. One can show that

$$\frac{\alpha_{1,\mathfrak{p}}}{\alpha_{2,\mathfrak{p}}} \notin \mu_\infty \qquad \forall' \mathfrak{p}\,.$$

$$\prod_{i=1}^{2} \det(1 - \theta' \otimes \chi(\mathrm{Frob}_{\mathfrak{p}})\alpha_{i,\mathfrak{p}} T) = \prod_{i=1}^{2} \det(1 \pm \theta(\mathrm{Frob}_{\mathfrak{p}})\alpha_{i,\mathfrak{p}} T) \qquad \forall' \mathfrak{p}\,.$$

One deduces

$$\det(1 - \theta' \otimes \chi(\mathrm{Frob}_{\mathfrak{p}}) T) = \det(1 \pm \theta(\mathrm{Frob}_{\mathfrak{p}}) T) \qquad \forall' \mathfrak{p}\,.$$

By Chebotarev, this means that $\theta' \otimes \chi$ and $\theta$ are locally quadratic twists.

By Ramakrishnan's theorem, there exists a quadratic character $\psi$ such that

$$\theta' \otimes \chi \simeq \psi \otimes \theta\,.$$

Hence

$$\varrho_{B,\ell} \simeq \theta' \otimes \chi \otimes \varrho \simeq \psi \otimes \theta \otimes \varrho \simeq \psi \otimes \varrho_{A,\ell}\,.$$

# Proof sketch of the Tate module tensor decomposition

When $A_{\overline{\mathbb{Q}}} \sim E^2$, we want to express $\varrho_{A,\ell} \simeq \theta \otimes \varrho$.

Let $\sigma \in G_K$. Then

$$^{\sigma}E^2 \sim {}^{\sigma}A_{\overline{\mathbb{Q}}} \sim A_{\overline{\mathbb{Q}}} \sim E^2$$

gives, by Poincaré, an isogeny $\mu_\sigma : {}^{\sigma}E \to E$.

Definition of $\varrho(\sigma)$:

$$\varrho(\sigma) : V_\ell(E) \xrightarrow{\sigma} V_\ell({}^{\sigma}E) \xrightarrow{\mu_\sigma} V_\ell(E) \xrightarrow{\cdot\alpha_\sigma} V_\ell(E)$$

for $\alpha_\sigma \in M^\times \subseteq \mathbb{Q}_\ell^\times$.

Definition of $\theta(\sigma)$:

$$\mathrm{Hom}(E, A_{\overline{\mathbb{Q}}}) \xrightarrow{\sigma} \mathrm{Hom}({}^{\sigma}E, A_{\overline{\mathbb{Q}}}) \xrightarrow{(\mu_s^{-1})^*} \mathrm{Hom}(E, A_{\overline{\mathbb{Q}}}) \xrightarrow{\cdot\alpha_\sigma^{-1}} \mathrm{Hom}(E, A_{\overline{\mathbb{Q}}}) .$$