Ordinary primes for some abelian varieties with extra endomorphisms

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Conference: Around Frobenius distributions and related topics
Organizers: Victoria Cantoral and Seoyoung Kim
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Notation

- k a number field.
- A an abelian variety defined over k of dimension $g \geq 1$.
- For a rational prime ℓ, let

$$T_\ell(A) := \varprojlim A[\ell^n](\overline{\mathbb{Q}}) \simeq \mathbb{Z}_\ell^{2g}\,, \qquad V_\ell(A) := T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\,.$$

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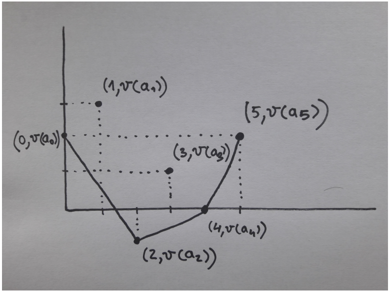
- Let $\mathfrak{p}$ be a "prime of k" of good reduction for A. We will denote by:
  - $A_\mathfrak{p}$ the reduction of A modulo $\mathfrak{p}$.
  - $P_\mathfrak{p}(A, T)$ the charpoly of $\mathrm{Frob}_\mathfrak{p}$ acting on $V_\ell(A)$ (for $\mathfrak{p} \nmid \ell$).
    (It has degree $2g$, integral coefficients, and is independent on ℓ).
  - $u_\mathfrak{p}(A)$ the number of roots of $P_\mathfrak{p}(A, T)$ in $\overline{\mathbb{Z}}_p$ which are p-adic units.
    (As $\alpha_\mathfrak{p} \cdot \overline{\alpha}_\mathfrak{p} = \mathrm{Nm}(\mathfrak{p})$ for any root $\alpha_\mathfrak{p}$, we have $0 \leq u_\mathfrak{p}(A) \leq g$.)

Newton polygons

- Let $P(T) = a_0T^n + a_1T^{n-1} + \cdots + +a_n \in \mathbb{Q}_p[T]$ with $a_0, a_n \neq 0$.
- Let v denote the extension to $\overline{\mathbb{Q}}_p$ of the p-adic valuation.
- The Newton polygon of $P(T)$ is the lower convex hull of the set

$$\{(i, v(a_i)) \,:\, i = 0, \ldots, n\}.$$



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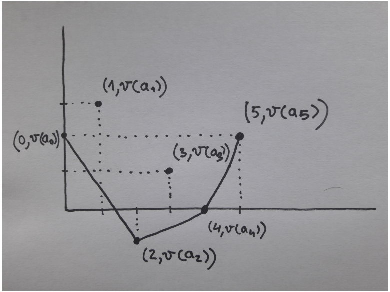


- For $s \in \mathbb{Q}$, the Newton polygon satisfies:

  x-length of the segment of slope $s = \#$roots of $P(T)$ of valuation s.

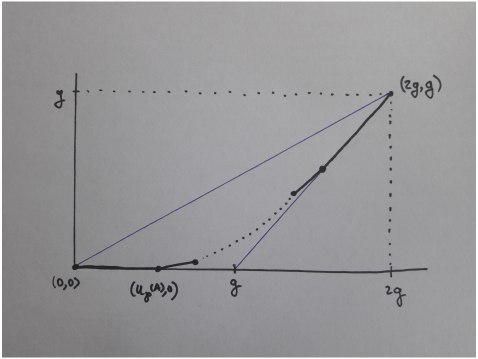
Newton polygons for abelian varieties

- $P_{\mathfrak{p}}(A, T) = a_0 T^{2g} + a_1 T^{2g-1} + \cdots + a_{2g-1} T + a_{2g}$ satisfies:

$$a_0 = 1, \quad a_{2g} = \mathrm{Nm}(\mathfrak{p})^g, \quad v(a_i) \geq 0, \quad v(a_{g+i}) = v(a_{g-i}\mathrm{Nm}(\mathfrak{p})^i) \geq i.$$

(v is normalized so that $v(\mathrm{Nm}(\mathfrak{p})) = 1$).

- $u_{\mathfrak{p}}(A) =$ number of roots of $P_{\mathfrak{p}}(A, T)$ of valuation 0.
- The Newton polygon of $P_{\mathfrak{p}}(A, T)$ looks like:

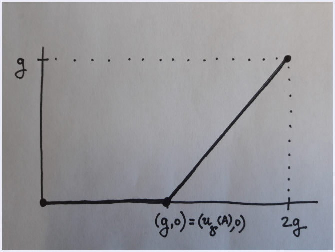


Ordinary primes

Definition/Proposition

$\mathfrak{p}$ is ordinary for A if any of the following equivalent conditions hold:

- $A_{\mathfrak{p}}[p](\overline{\mathbb{F}}_p)$ has cardinality p^g.
- $u_{\mathfrak{p}}(A) = g$.
- The central coefficient a_g of $P_{\mathfrak{p}}(A, T)$ is not divisible by p.
- The Newton polygon of $P_{\mathfrak{p}}(A, T)$ looks like:



The conjecture

- Let $\mathrm{P}_{\mathrm{ord}}(A)$ denote the set of primes of k which are ordinary for A.

Conjecture (Often attributed to Serre)

For every abelian variety A/k, the set $\mathrm{P}_{\mathrm{ord}}(A)$ has a nonzero density.

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- Suppose $\mathfrak{p}$ is of absolute residue degree 1 (i.e., $\mathrm{Nm}(\mathfrak{p}) = p$ is prime).
- If A is an elliptic curve, then $|a_1| \leq 2\sqrt{p}$ and thus

$$p \mid a_1 \quad \Rightarrow \quad a_1 = 0 \quad \text{if } p \geq 5.$$

- If A is an abelian surface, then $-2p \leq a_2 < 6p$ and thus

$$p \mid a_2 \quad \Rightarrow \quad a_2 = -2p, -p, 0, p, \ldots, 5p\,.$$

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- The number of values of a_g for which $\mathfrak{p}$ can fail to be ordinary for A:
  - Stays bounded as p grows if $g \leq 2$.
  - Grows arbitrarily large if $g \geq 3$.

Ordinary primes in context

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Ordinary primes in context

- The notion of ordinary can be defined in much greater generality (it can essentially be defined for any arithmetico-geometric object).
- The notion of "ordinary" appears in:
  - The construction of p-adic L-functions.
  - The construction of p-adic families (Hida theory).
  - The *abundance* of ordinary primes for an abelian surface is a technical assumption in an automorphic lifting theorem of **Boxer**-**Calegari**-**Gee**-**Pilloni**.

    (This ALT leads to the meromorphicity of the Hasse-Weil L-function of a generic abelian surface defined over a totally real field).
  - The *abundance* of ordinary primes can be used to establish the average running time of modulo p point counting algorithms for curves C over number fields.

    (First compute $P_{\mathfrak{p}}(\mathrm{Jac}(C), T) \pmod{p}$. Then find the lift to $\mathbb{Z}[T]$).

Results for $g \leq 2$

Theorem (Serre; 1981)

Let A/k be an elliptic curve. Then $\mathrm{P}_{\mathrm{ord}}(A)$ has density:

- 1 if $A_{\overline{\mathbb{Q}}}$ does no have CM.
- $\frac{1}{[kF:k]}$ if $A_{\overline{\mathbb{Q}}}$ has CM by an imaginary quadratic field F.

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Let A/k be an abelian surface. Then $\mathrm{P}_{\mathrm{ord}}(A)$ has density:

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- $\frac{1}{[kF^*:k]}$ if $A_{\overline{\mathbb{Q}}}$ has CM by a quartic CM field F.
- $\frac{1}{[kF_1F_2:k]}$ if $A_{\overline{\mathbb{Q}}} \sim E_1 \times E_2$, at least one E_i is CM, and $F_i = \mathrm{End}(E_i) \otimes \mathbb{Q}$.

In particular, $\mathrm{P}_{\mathrm{ord}}(A)$ has density 1, $1/2$, or $1/4$.

Idea behind the proof

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In particular, $\mathrm{P}_{\mathrm{ord}}(A)$ has density 1 or $1/2$.

- Consider the ℓ-adic representation $\varrho_{A,\ell}\colon G_k \to \mathrm{Aut}(V_\ell(A))$.
- Let G_ℓ be the Zariski closure of the image of $\varrho_{A,\ell}$.
- Then **Serre** shows:

$$\mathrm{Dens}(\mathrm{P}_{\mathrm{ord}}(A)) = 1 - \frac{\#\text{conn. comp. of } G_\ell \text{ on which } \mathrm{Tr}(\varrho_{A,\ell}) \equiv 0}{\#\text{conn. comp. of } G_\ell}.$$

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- Let $\chi_\ell \colon G_k \to \mathbb{Q}_\ell^\times$ be the ℓ-adic cyclotomic character.
- **Sawin** shows:

$$\mathrm{Dens}(\mathrm{P}_{\mathrm{ord}}(A)) = 1 - \frac{\#\text{conn. comp. of } G_\ell \text{ on which } \mathrm{Tr}(\wedge^2 \varrho_{A,\ell} \otimes \chi_\ell^{-1}) \equiv \text{const.}}{\#\text{conn. comp. of } G_\ell}.$$

In fact, here "const." means -2,-1,0,1,2.

The ℓ-adic method of Katz, Ogus, and Serre

- E a number field.
- λ a prime of E lying above ℓ.
- S a finite set of places of k.
- Let $\varrho : G_k \to \mathrm{Aut}(E_\lambda^d)$ a d-dim. cont. rep. unramified outside S.

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- Let $\varrho : G_k \to \mathrm{Aut}(E_\lambda^d)$ a d-dim. cont. rep. unramified outside S.
- We say that ϱ is:
  - $\mathcal{O}_E$-integral if $\mathrm{CharPoly}(\varrho(\mathrm{Frob}_\mathfrak{p}))$ lies in $E[T] \subseteq E_\lambda[T]$ for all $\mathfrak{p} \notin S$.
  - of weight w if it is integral and every root of $\mathrm{CharPoly}(\varrho(\mathrm{Frob}_\mathfrak{p}))$ is a $\mathrm{Nm}(\mathfrak{p})^w$-Weil number for all $\mathfrak{p} \notin S$.

(Recall: $\alpha \in \overline{\mathbb{Q}}$ is a q-Weil number if $|\iota(\alpha)| = \sqrt{q}$ for all $\iota \colon \mathbb{Q}(\alpha) \hookrightarrow \mathbb{C}$).

The ℓ-adic method of Katz, Ogus, and Serre

Proposition (Katz)

Suppose that ϱ is $\mathcal{O}_E$-integral of weight 1. There exists a finite extension k'/k and a set R of primes of k' such that:

- Every $\mathfrak{p}$ in R is of absolute residue degree 1 (write $\mathrm{Nm}(\mathfrak{p}) = p$).
- $p \nmid \mathrm{Tr}(\varrho(\mathrm{Frob}_{\mathfrak{p}}))$ for every $\mathfrak{p}$ in R.
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Proposition (Ogus+ε)

Suppose that ϱ is $\mathcal{O}_E$-integral of weight 1 and $d = \dim(\varrho) \geq 3$. There exists a finite extension k'/k and a set R of primes of k' such that:

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The conjecture for elliptic curves and abelian surfaces

Corollary

$\underline{\mathrm{Dens}}(\mathrm{P}_{\mathrm{ord}}(A)) > 0$ if A/k is an elliptic curve.

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Katz' proposition applied to $\varrho = \varrho_{A,\ell} \Rightarrow \mathrm{Dens}(\mathrm{P}_{\mathrm{ord}}(A_{k'})) = 1$ for some finite $k'/k \Rightarrow \underline{\mathrm{Dens}}(\mathrm{P}_{\mathrm{ord}}(A)) > 0$.

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Remark

What is known about $\underline{\mathrm{Dens}}(\mathrm{P}_{\mathrm{ord}}(A)) > 0$ in higher dimension?

- Partial results by **Suh** when A is a Hilbert-Blumenthal ab. var.
- Known when $\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathbb{Z}$ and $\mathrm{MT}(A)$ is “small", by **Pink**.
- Known if A has potential CM.

Proposition (Katz)

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- May assume $\varrho : G_k \rightarrow \mathrm{Aut}(\mathcal{O}_{E_\lambda}^d)$.
- Let n be such that $\lambda^n \nmid d$ and let $\overline{\varrho}$ be the reduction of $\varrho \pmod{\lambda^n}$.
- Choose k'/k finite such that $\overline{\varrho}|_{G_{k'}} = 1$.
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- $R =$ primes of k' of abs. res. degree 1, of good red for A, not above ℓ, and $> d^2$. Claim: Any $\mathfrak{p}$ in R satisfies (2).
- Then $a_{\mathfrak{p}} := \mathrm{Tr}(\varrho(\mathrm{Frob}_{\mathfrak{p}})) \equiv d \pmod{\lambda^n}$ and in particular $a_{\mathfrak{p}} \neq 0$.
- If $a_{\mathfrak{p}} = p b_{\mathfrak{p}}$ for some $b_{\mathfrak{p}} \in \mathcal{O}_E$, then

$$|\mathrm{Nm}_{E/\mathbb{Q}}(b_{\mathfrak{p}})| = \frac{|\mathrm{Nm}_{E/\mathbb{Q}}(a_{\mathfrak{p}})|}{p^{[E:\mathbb{Q}]}} = \left(\frac{d}{\sqrt{p}}\right)^{[E:\mathbb{Q}]} < 1\,.$$

λ-adic representations attached to abelian varieties

- Suppose that A/k is such that:

$$E \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}\,.$$

- We will consider

$$V_\lambda(A) := V_\ell(A) \otimes_{E\otimes\mathbb{Q}_\ell} E_\lambda \quad \text{w.r.t} \quad E \otimes \mathbb{Q}_\ell \simeq \prod_{\lambda'|\ell} E_{\lambda'} \to E_\lambda\,.$$

- $V_\lambda(A)$ has dimension $2g/[E:\mathbb{Q}]$ over E_λ.

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- This gives rise to the λ-adic representation attached to A

$$\varrho_{A,\lambda}\colon G_k \to \mathrm{Aut}(V_\lambda(A))\,.$$

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- It is $\mathcal{O}_E$-integral, of weight 1, and $V_\ell(A) \simeq \bigoplus_{\lambda|\ell} V_\lambda(A)$.
- $S =$ finite set containing the primes of bad reduction for A.
- For $\mathfrak{p}$ outside S_ℓ, set $P_\mathfrak{p}(V_\lambda(A), T) := \mathrm{CharPoly}(\varrho_{A,\lambda}(\mathrm{Frob}_\mathfrak{p}))$.
- $P_\mathfrak{p}(A, T) = \prod_{\lambda|\ell} P_\mathfrak{p}(V_\lambda(A), T)$.

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- There exists E with $[E : \mathbb{Q}] = 2g$ such that

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- WLOG, we may assume that $E \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$.
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- WLOG, we may assume that $E \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$.
- Let $\mathfrak{p}$ be of abs. deg. 1 and totally split in E. Claim: it is ordinary.
- Choose ℓ totally split in E, so that $V_\lambda(A)$ has dimension 1 over $\mathbb{Q}_\ell$.
- The roots of $P_{\mathfrak{p}}(A,T)$ are $a_{\lambda,\mathfrak{p}} = \varrho_{A,\lambda}(\mathrm{Frob}_{\mathfrak{p}}) \in \mathcal{O}_E$ for $\lambda \mid \ell$.
- By Weil's theorem:

$$a_{\lambda,\mathfrak{p}} \cdot a_{\overline{\lambda},\mathfrak{p}} = p = \mathrm{Nm}(\mathfrak{p})\,.$$

- As $a_{\lambda,\mathfrak{p}}, a_{\overline{\lambda},\mathfrak{p}} \in \mathcal{O}_E$, we have $\{v(a_{\lambda,\mathfrak{p}}), v(a_{\overline{\lambda},\mathfrak{p}})\} = \{0,1\}$.
  (v an extension to $\overline{\mathbb{Z}}$ of the $\mathfrak{P}$-adic valuation for some $\mathfrak{P} \mid p$ of E).

New results in dimension 3

Theorem 1 (F.)

Let A/k be an abelian threefold such that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ contains an imaginary quadratic field. Then $\underline{\mathrm{Dens}}(\mathrm{P}_{\mathrm{ord}}(A)) > 0$.

Theorem 2 (F.)

Let A/k be an abelian threefold for which there exists F/k such that:

- $\mathrm{End}(A_F) \otimes \mathbb{Q}$ contains a totally real cubic field E.
- F does not contain E.

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- It applies to the Jacobian of a Picard curve:

$$y^3 = f(x)\,, \qquad \text{for some separable } f(x) \in k[x] \text{ of degree 4.}$$

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Then $\underline{\mathrm{Dens}}(\mathrm{P}_{\mathrm{ord}}(A)) > 0$.

- Let f be a classical newform of weight 2 and cubic coefficient field. The Thm. applies to the $A_f/\mathbb{Q}$ attached to f by Eichler–Shimura.

Theorem 3 (F.)

Let A/k be an abelian fourfold such that:

- $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ contains an imaginary quadratic field E.
- The pair (E, A) has signature $(2, 2)$.

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- The signature of (E, A) is the tuple $(r_\tau)_{\tau\colon E \hookrightarrow \mathbb{C}}$ defined by:

$$r_\tau = \dim_{\mathbb{C}}(H^0(A_{\mathbb{C}}, \Omega^1_{A_{\mathbb{C}}/\mathbb{C}}) \otimes_{E\otimes\mathbb{C},\tau} \mathbb{C}),$$

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  "the multiplicity of the action of E on $H^0(A_{\mathbb{C}}, \Omega^1_{A_{\mathbb{C}}/\mathbb{C}})$ via τ".
- The theorem applies when:
  - $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} = E$ and the signature is $(2, 2)$ (A is of Albert type IV).
  - $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ is a quaternion algebra (i.e. A has Albert type II or III). That is, A is a so-called "fake abelian surface".

Proof of Theorem 1

Theorem 1 (F.)

Let A/k be an abelian threefold such that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ contains an imaginary quadratic field. Then $\underline{\mathrm{Dens}}(\mathrm{P}_{\mathrm{ord}}(A)) > 0$.

Proof of Theorem 1

Theorem 1 (F.)

Let A/k be an abelian threefold such that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ contains an imaginary quadratic field. Then $\underline{\mathrm{Dens}}(\mathrm{P}_{\mathrm{ord}}(A)) > 0$.

- WLOG, we may assume $E \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Q}$ and $E \subseteq k$.
- Choose ℓ split in E, so that $V_\ell(A) \simeq V_\lambda(A) \oplus V_{\overline{\lambda}}(A)$.
- By **Ogus'** proposition, there exist k'/k and a set of primes R s.t.:

$$p := \mathrm{Nm}(\mathfrak{p}) \nmid b_{\mathfrak{p},\lambda} := \mathrm{Tr}(\wedge^2 \varrho_{A,\lambda}(\mathrm{Frob}_\mathfrak{p})),\ \forall \mathfrak{p} \in R \quad \text{and} \quad \mathrm{Dens}(R) = 1\,.$$

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- Let v be an ext. to $\overline{\mathbb{Z}}_p$ of the $\mathfrak{P}$-adic valuation for some $\mathfrak{P} \mid p$ of E.
- One among $v(b_{\mathfrak{p},\lambda})$ and $v(b_{\mathfrak{p},\overline{\lambda}})$ must be 0.
- Let α, β, γ be the roots of $P_{\mathfrak{p}}(V_\lambda(A), T)$.
- We may assume that $v(\alpha), v(\beta), v(\gamma)$ is $0, 0, 1$.

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- Note that the roots of $P_\mathfrak{p}(V_{\overline{\lambda}}(A), T)$ are p/α, p/β, p/γ.

A mild generalization

Let A/k be such that $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ contains a simple $\mathbb{Q}$-algebra D.

Let K denote the center of D and t the Schur index of D.

Corollary (F.)

Let A/k be as above. If K is imaginary quadratic and either:

1. $t = g/2$ and $g \mid 4$; or
2. $t = g/3$ and $g \mid 9$; or
3. $t = g/4$ and $g \mid 16$ and (A, K) has signature $(g/2, g/2)$;

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$$P_{\mathfrak{p}}(A, T) = (Q(T) \cdot \overline{Q(T)})^t \,, \qquad \text{where } Q(T) \in \mathcal{O}_K[T].$$

- The proof then reduces to

$$\begin{cases} \text{the case of abelian surfaces} & (1). \\ \text{the situation of Theorem 1} & (2). \\ \text{the situation of Theorem 3} & (3). \end{cases}$$

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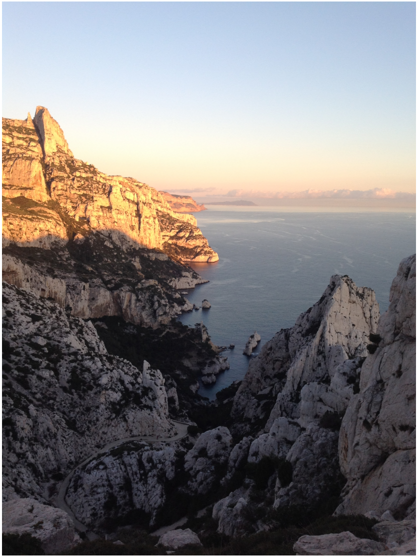
- Alternative proof that $\underline{\mathrm{Dens}}(\mathrm{P}_{\mathrm{ord}}(A)) > 0$ if A is a “fake abelian surface".



Thank you for your attention!