Serre's open image theorem plan

Pip Goodman

Below is the plan to prove Serre's open image theorem. I've somewhat modified Serre's original argument to make it easier. In particular, this includes using Falting's Isogeny Theorem, Raynaud's Theorem and replacing Serre's Tori with algebraic Hecke characters.

Feel free to email me if you have any questions pip.goodman@ub.edu.

First talk

See here for a brief summary of Class Field Theory. This talk should roughly cover the material in Sections 1.1 to 1.5 of Serre's paper [Ser72]. Here's a rough plan including the essentials:

- Recall main statements of local class field theory.
- Global class field theory in terms of the adeles. Also ray class groups in terms of the adeles (finiteness is what's important here really).
- Explicit examples of ray class groups using CFT.
- Introduce fundamental characters (see §1.3 [Ser72]).
- Prove [Ser72, Prop. 3, pg 265].

Second talk

The plan of this talk is essentially to replace the content covered in Serre's §1.8 to §2.7 with the statement of a couple of theorems and then introduce compatible systems of ℓ-adic representations and prove a basic result about them.

- State Raynaud's Theorem (see [ALS16, Thm. 4] for it specialised to abelian varieties).
- Make Raynaud's Theorem explict for elliptic curves (use determinant = cyclotomic character = fundamental character of level 1).

- State classification of maximal subgroups of $\mathrm{GL}_2(\mathbb{F}_\ell)$. Google gave me this reference, but there's plenty available. The most accurate (but also perhaps most complicated) is [BHRD13].
- Deduce the image of Galois can only be contained in subgroups whose size doesn't grow with ℓ for only finitely many ℓ.
- Define compatible systems of ℓ-adic representations [Ser68, I-10].
- Prove [Ser68, I-10, Theorem] (a unicity statement about semisimple compatible systems).

Third talk

Here we are essentially following an alternative way to cover the material in Serre's §3.1–3.4, along with examples.

- Define algebraic Hecke characters and their ℓ-adic avatars (see [Sch88, Chp. 0, §5]).
- Define algebraic homomorphisms and show which ones arise as the infinity type of an algebraic Hecke character (see [Sch88, Chp. 0 §2, 3]).
- Give explicit examples of algebraic Hecke characters. Of particular relevance here are the ones which arise from CM elliptic curves (see [Sil94, §II, Thm. 9.2, pg 168]).

Fourth talk

In an effort to simplify the material we need to cover, I've removed Serre's Tori from the picture, which he spends a lot of time on. These are essentially just Serre's way of reformulating algebraic Hecke characters so that they become characters of certain algebraic groups. This is explained in Chapter 0 §7 of [Sch88].

Here's the basic dictionary for Serre's Tori T, T_m, S_M via their character groups $X(T), X(T_m), X(S_m)$.

- $X(T)$ algebraic homomorphisms.
- $X(T_m)$ algebraic homomorphisms which are infinity types of algebraic Hecke characters.
- $X(S_m)$ algebraic Hecke characters.

The results you will prove in this talk (listed below) are the essential step in the paper towards proving the main theorem. Essentially they'll allow us to show that if there are infinitely many reducible mod ℓ representations, then they must come from an algebraic Hecke character.

This material essentially covers §3.5 & §3.6 of Serre's article.

- Prove [Ser72, Prop. 20, pg 289].
- Prove [Ser72, Thm. 1, pg 291].

Fifth talk

This talk focuses on ellitpic curves. You'll prove that the Tate module V_ℓ of an elliptic curve is irreducible and show that multiplicative reduction gives rise to transvections in the image of the mod ℓ representations.

- Prove the Irreducibility Theorem [Ser68, IV-9,§2.1] (you'll need to at least state the theorem and corollary on page IV-7).
- State the basics of Tate curves [Sil94, pg 422] to let you prove [Sil94, pg447, Cor. 6.2].

Sixth talk

In this talk you'll need to develop some background from ℓ-adic Lie groups and Lie algebras and use it prove the ℓ-adic images are open. One reference for this is [Hoo42].

- Show the image of Galois on V_ℓ is a Lie group (use that the image is closed).
- Prove [Ser68, IV-11,Theorem] using Falting's Isogeny Theorem. Serre explains a bit about this in his paper [Ser66]. (Ribet also does this in his paper [Rib76] for a larger class of abelian varieties.)

Seventh talk

State Theorem 2 and prove it (proof of surjecting at all but finitely many primes mod ℓ). Sections 4.1 and 4.2 of [Ser72].

Eighth talk

The purpose of this talk is to give some explicit examples and make a start on why the image is actually open in $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$.

- Explicit examples (see e.g. Section 5 of Serre's article).
- State Theorem 3 in [Ser72], along with the Proposition and main lemma on page IV-19 of [Ser68].
- Prove Lemma 3 on page IV-23 [Ser68]. (This essentially shows that for $\ell > 3$ surjecting mod ℓ is equivalent to surjecting ℓ-adically.)

Ninth talk

Finish the proof that the adelic image is open. The essence of this talk is in proving that the images for the individual ℓ's do not 'mix' via a Goursat's Lemma type argument.

- Finish the proof of the main lemma. Pages IV-24 to IV-27.

References

[ALS16] Samuele Anni, Pedro Lemos, and Samir Siksek. Residual representations of semistable principally polarized abelian varieties. *Res. Number Theory*, 2:Paper No. 1, 12, 2016.

[BHRD13] John N. Bray, Derek F. Holt, and Colva M. Roney-Dougal. *The maximal subgroups of the low-dimensional finite classical groups*, volume 407 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2013. With a foreword by Martin Liebeck.

[Hoo42] Robert Hooke. Linear p-adic groups and their Lie algebras. *Ann. of Math. (2)*, 43:641–655, 1942.

[Rib76] Kenneth A. Ribet. Galois action on division points of Abelian varieties with real multiplications. *Amer. J. Math.*, 98(3):751–804, 1976.

[Sch88] Norbert Schappacher. *Periods of Hecke characters*, volume 1301 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[Ser66] Jean-Pierre Serre. Groupes de Lie l-adiques attachés aux courbes elliptiques. In *Les Tendances Géom. en Algèbre et Théorie des Nombres*, volume No. 143 of *Colloq. Internat. CNRS*, pages 239–256. CNRS, Paris, 1966.

[Ser68] Jean-Pierre Serre. *Abelian l-adic representations and elliptic curves.* McGill University lecture notes written with the collaboration of Willem Kuyk and John Labute. W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[Ser72] Jean-Pierre Serre. Propriétés galoisiennes des points d'ordre fini des courbes elliptiques. *Invent. Math.*, 15(4):259–331, 1972.

[Sil94] Joseph H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.