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Universitat Politècnica de Catalunya
Universitat Autònoma de Barcelona
Centre de Recerca Matemàtica
Universitat de Barcelona

BARCELONA NUMBER THEORY STUDY GROUP

RATIONAL POINTS ON CURVES OF HIGH GENUS

Reference person: Roberto Gualdi, roberto.gualdi@upc.edu

Venue: Tuesdays 15h20 – 16h50, room iA at UB

The search for rational points on algebraic varieties is one of the main themes in number theory, as witnessed by the vibrant activity of the community on problems like the Birch–Swinnerton-Dyer conjecture, the Bombieri–Lang conjecture and the validity or failure of the local-global principle.

In practise, given a smooth projective variety X defined over $\mathbb{Q}$, it is desirable to know as much as possible about the set $X(\mathbb{Q})$ of its ($\mathbb{Q}$-)rational points. Questions like "Is this set finite? Infinite? Dense? Empty?" or "Where are the rational points located on X?" have been the origin of many spectacular results in the last century, and keep inspiring researchers today.

A convenient starting point for such questions is the case in which X is a *smooth projective curve*. Despite the restrictedness on the dimension, this instance is far from being trivial and already contains many interesting situations, including the one of Fermat's Last Theorem.

It has turned out that in this case the cardinality of $X(\mathbb{Q})$ is very much influenced by the topological nature of X, and more specifically by its *genus*. In fact, if X has genus 0 and it has at least one rational point, then it is isomorphic to the projective line, and therefore it contains infinitely many of them. In the case of genus 1, the Mordell–Weil theorem ensures that $X(\mathbb{Q})$ is a finitely generated group, and hence its finiteness is equivalent to the vanishing of its rank. Finally, when X has genus at least 2, it was conjectured by Mordell in 1922 that $X(\mathbb{Q})$ is finite.

Mordell's conjecture was solved by Faltings in 1984 [Fal83, Fal84], using techniques from Arakelov geometry. Faltings's arguments rely on the consideration of a new notion of height function for abelian varieties, and pass through the proof of the Shafarevich's conjecture on the finiteness of isomorphism classes of polarized abelian varieties over $\mathbb{Q}$ with prescribed dimension, degree and set of primes of bad reduction. ***Despite its intrinsic interest and beauty, Faltings's proof is beyond the scope of this seminar***.

Later proofs of Mordell's conjecture were given by Vojta [Voj91], using techniques of diophantine approximation, and more recently by Lawrence and Venkatesh [LV20], using p-adic Hodge theory.

The goal of this seminar is to study the diophantine proof of Faltings's celebrated theorem, following the arguments of Vojta, as simplified by Bombieri in [Bom90]**.**

In order to achieve this goal, we will need to start by collecting some basic material about height functions and their theory on abelian varieties. After a small but useful digression on the Mordell–Weil theorem we will then focus on the case of Jacobians and develop the details of Vojta–Bombieri's proof of Faltings's theorem. Our primary reference for this main part of the seminar will be the brilliant book by Bombieri and Gubler [BG06]. The willing reader is also referred to corresponding sections of the book by Hindry and Silverman [HS00] for an alternative and excellent source.

If the participants are willing to do so, the seminar will end with a peek on very recent advances on the questions of determining the number of rational points on a curve of genus at least 2. More precisely, a combination of [DGH21] and of [Küh21] has affirmatively answered a question by Mazur [Maz86], leading to the proof of the fact that

$$\#X(\mathbb{Q}) \leq c_g^{1+\operatorname{rank} \operatorname{Jac}(X)(\mathbb{Q})},$$

where c_g is a constant only depending on the genus of the curve X. This bound is known as the *uniform Mordell–Lang conjecture*, and it is proved by combining a height inequality "in families" with a uniform Bogomolov-type result. This part last of the seminar will be studied by referring to the two mentioned references, and to [Gao21].

A USER GUIDE

Every section of these notes concerns a talk for the learning seminar and, after a quick presentation of the main subject of the lecture, it features a **"Condensed abstract"** and a **"Tips"** paragraph. The first one is intended to be a one-sentence motto for the lecture, containing the primary goal that the speaker should aim to for a successful talk. The second one is a collection of suggestions intended to guide the speaker in the preparation of their exposition: as long as the goal is accomplished and the time schedule respected (1h30 per talk), the final choice of propositions, examples and proofs to present is left to their taste, as well as the depth of details to enter.

Also, the participants are free to choose their favourite references among the one proposed above (or further ones!); however, for a better global coordination, it is suggested to stick to the notation of [BG06]. In any case, the organizer stays at disposal for discussions on how to organize any specific lecture or for any doubts concerning the outlines of the talks given here.

This document is available online at `https://roberto-gualdi.staff.upc.edu/documents/BNTSG24_program.pdf`.

Contents

Welcome session (27.02, *Roberto Gualdi*)

The goal of this first meeting is to give a very vague presentation of the study group, sum up the contents of the talks and agree on a planned schedule among participants by finding an answer to the question "*Who talks when?*". Whoever is interested in participating to the learning seminar is heartily invited and will be warmly welcomed. However, for a fruitful participation to the lectures, some previous knowledge in algebra, algebraic geometry and number theory is recommended.

◀ Preliminaries on Néron–Tate heights ▶

1. The height machine (12.03, *Javier Guillán*)

The aim of this first talk is to introduce and study a major and ubiquitous tool in number theory and arithmetic geometry: the one of *Weil heights* on projective varieties. Having at disposal the usual height on a projective space, any choice of a Cartier divisor on a projective variety defines local and global heights for their points. For Cartier divisors in the same principal equivalence class, the resulting height functions only differ by a bounded term, so that one can associate to every line bundle on a projective variety a unique height function (up to a bounded function).

One fundamental property of the height function associated to an ample line bundle is *Northcott's theorem*, claiming the finiteness of the set of points with bounded degree and height on the variety.

Condensed abstract: the notion of Weil height associated to a line bundle on a projective variety is introduced, and Northcott's theorem for them is explained.

Tips: the idea is to explain the Weil's height machine from [HS00, Theorem B.3.2]. It would be good to integrate it with [BG06, 2.2, 2.3 and 2.4], with a special focus on [BG06, Theorem 2.3.8] and of [BG06, Theorem 2.4.9].

2. Néron–Tate heights on abelian varieties (19.03, *Ignasi Sánchez*)

Recall from the previous talk that any choice of a line bundle on a projective variety defines a height function on its points only up to a constant function. In the case of abelian varieties, one can remove this undeterminacy by a smart limit argument.

This procedure yields a well-defined height function associated to every line bundle of the abelian variety, in a way which is compatible with the group structure. The obtained function is called the *Néron–Tate height*, and it induces a bilinear form on the space of algebraic points of the abelian variety.

Condensed abstract: after a quick recall on abelian varieties and their line bundles, the construction of Néron–Tate heights on them and of the associated bilinear form is explained.

Tips: the speaker should give an account of [BG06, 9.2] and of [BG06, sections from 9.3.4 to 9.3.10]. In particular, they should focus on the proof of [BG06, Theorem 9.2.7] and of [BG06, Theorem 9.3.5], together with a sum-up of its corollaries. The corresponding part in Hindry–Silverman's book can be taken from [HS00, B.5].

◄ The Mordell–Weil theorem ►

3. The weak Mordell–Weil theorem for elliptic curves (02.04, *Eloi Torrents*)

It is a key fact in the proof of Mordell's conjecture that the group of rational points of an abelian variety is finitely generated. This statement, known as the *Mordell–Weil theorem*, was proved by Weil in his thesis. Its classical proof consists in two steps: first one proves the finiteness of a certain quotient group, and then one uses a Fermat descent argument.

The goal is this lecture is to present a naive proof of the first step of the proof, called the *weak Mordell–Weil theorem*, in the case of an elliptic curve.

Condensed abstract: the weak Mordell–Weil theorem for elliptic curves is proven in a naive way.

Tips: the speaker should present the content of [BG06, 10.2], which is centered around the proof of [BG06, Theorem 10.2.14].

4. The Mordell–Weil theorem for abelian varieties (09.04, *Michele Fornea*)

A less naive proof of the weak Mordell–Weil theorem uses a generalization of Kummer theory to abelian varieties, and involves the use of the Chevalley–Weil theorem. Once the weak version of the theorem for abelian varieties is established, deducing the full Mordell–Weil theorem from it is just a matter of Fermat descent via a tricky use of the Néron–Tate height function.

Condensed abstract: after having proven the weak Mordell–Weil theorem using Kummer theory, the Mordell–Weil theorem for abelian varieties is deduced.

Tips: the speaker should cover [BG06, 10.4 and 10.6], aiming at [BG06, Theorem 10.4.1 and

Theorem 10.6.1]. In particular, they can use without proof the Chevalley–Weil theorem from [BG06, Corollary 10.3.13].

The content of [HS00, introduction of part C and section C.1] is another excellent reference for the preparation of the talk.

◀ Faltings's theorem (after Vojta and Bombieri) ▶

5. Néron–Tate heights on Jacobians (16.04, *Enric Florit*)

As the main theorem of the seminar is concerned with curves, it is no surprise that one extra ingredient that we will need is the consideration of heights on special abelian varieties, that is for Jacobians. In this case, the theta divisor offers a canonical choice for the Néron–Tate height, inducing a symmetric positive semidefinite bilinear form on the space of algebraic points of the Jacobian.

Such a *Néron–Tate bilinear form* appears in the statement of two useful results, Mumford's formula and Mumford's gap principle. In particular, the second one affirms that rational points of a curve of genus at least 2 have sufficiently distant norm with respect to the above bilinear form. This will be key in Vojta–Bombieri's proof of Mordell's conjecture.

Condensed abstract: after recalling the notion of Jacobian of a curve and of its theta divisor, the bilinear form associated to the corresponding Néron–Tate height is introduced and studied.

Tips: the speaker should first recall the notion of Jacobian of a curve and of its theta divisors, referring for instance to [BG06, 8.10] and avoiding giving proofs. After that, they should give an account of [BG06, 9.4], aiming at the proof of Mumford's formula [BG06, Proposition 9.4.3] and Mumford's gap principle [BG06, Theorem 9.4.14].

6. The geometric part – Vojta divisors (30.04, *Santi Molina*)

With this lecture we officially inaugurate our detailed study of the proof of Mordell's conjecture. The first step towards this goal is to make some geometric considerations and constructions from a curve C defined over an arbitrary field.

In particular, one introduces a family of divisors, called *Vojta divisors*, on the surface $C \times C$. The study of the corresponding intersection numbers allows to decompose any Vojta divisor V into the difference of two very ample divisors; this will allow the calculation of the associated height function. Finally, Mumford's formula from the previous talk can be adapted to the height function defined from V.

Condensed abstract: the family of Vojta divisors on the square of a curve is defined, and a version of Mumford's formula for them is proven.

Tips: the speaker should explain [BG06, 11.2 and 11.3]. The main result to prove is [BG06, Proposition 11.3.1].

7. The analytic part – A bound for the Néron–Tate bilinear form (07.05, *Xevi Guitart*)

If the previous talk was more of geometric nature, this one has a much more analytic flavour. Its goal is, given a Vojta divisor V satisfying certain conditions, to provide some meaningful lower bounds for the Néron–Tate bilinear form associated to V, in terms of the Néron–Tate height of its arguments.

In doing so, one uses the local version of a theorem by Eisenstein giving upper bounds for the absolute values of the coefficients of the Taylor expansion of an algebraic function in one variable.

Condensed abstract: a lower bound for the Néron–Tate bilinear form associated to a Vojta divisor is proven, using a local version of Eisenstein theorem.

Tips: the speaker should cover [BG06, 11.4 and 11.6]. The two main result of the talk are [BG06, Lemma 11.6.5] and [BG06, Lemma 11.6.7], the latter of which uses the local Eisenstein theorem from [BG06, Theorem 11.4.1]. It would be good to present the proofs of these three mentioned results, or at least to indicate the main strategies for them.

8. The diophantine part – A Vojta divisor of small height (14.05, *Marc Masdeu*)

The last preparative lecture before the final proof has a diophantine character. The main idea is to construct a divisor which is principally equivalent to a Vojta divisor V, and such that its height (in a sense to make precise) is small. Achieving this goal requires the use of geometry of numbers, the thorough study of which is beyond the scope of this seminar.

The obtained divisor has a distinguished section which can be projected to $\mathbb{P}^1 \times \mathbb{P}^1$ and interpreted there as a bihomogeneous polynomial. In the same spirit as before, this bihomogeneous polynomial can be shown to be of small height, as an application of Roth's lemma.

Condensed abstract: using the Riemann–Roch theorem and Siegel's lemma, a nonzero global section with small height is constructed.

Tips: it is suggested that the speaker aims at the main ideas of the proof of [BG06, Lemma 11.7.3], without explaining all the details. In particular, it would be important to stress how the Riemann–Roch theorem and Siegel's lemma enter the proof. Then, they should at least state [BG06, Lemma 11.8.6], and clarify how this is connected with [BG06, Lemma 11.7.3] and Roth's lemma.

9. Piecing up – Vojta's inequality and Faltings's theorem (21.05, *Francesc Fité*)

With the results of the previous talks, there is only one ingredient missing for our final proof: *Vojta's inequality*, which assert that any two points of a curve having large and largely different Néron–Tate heights must form a sufficiently large angle with respect to the Néron–Tate bilinear form.

The proof of Mordell's conjecture follows now by combining the freshly obtained Vojta's inequality with the Mordell–Weil theorem, Northcott's theorem and Mumford's gap principle.

Condensed abstract: after the proof of Vojta's inequality, Faltings's theorem is proven.

Tips: the speaker should cover the material of [BG06, 11.9]. The two main results to explain in details are Vojta's theorem [BG06, Theorem 11.9.1] and [BG06, Theorem 11.1.1].

◄ A uniform Mordell–Lang statement ►

10. Betti map, height inequality and equidistribution (28.05, *Martín Sombra*)

Knowing that the set of rational points on a curve of sufficiently high genus is finite opens several further questions. For instance, one can ask whether the number of such points can be bounded solely in terms of the genus of the curve. This ambitious goal seems still a bit out of reach; however, it was recently shown that a bound in terms of the genus *and* of the Mordell–Weil rank of the Jacobian of the curve can be given.

We start in this talk an account for the proof of this result, known as the *uniform Mordell-Lang conjecture*. Two fundamental ingredients of the proof are the *Betti map*, coming from locally identifying the fibers of an abelian scheme over a base with a torus, and the *Betti form*, which is a closed semipositive $(1,1)$-form on the analytification of the abelian scheme.

These two notions are fundamental to define so-called *nondegenerate subvarieties* of the abelian scheme, to show that generic subvarieties are such, and to prove that outside of the non-degeneracy locus the height of a point can be uniformly compared with the one of its projection to the base. This last claim is the content of the so-called *height inequality*, proved by Dimitrov, Gao and Habegger.

One last ingredient that will be needed is the *equidistribution theorem*, in the form given by Kühne, which describes the asymptotic behaviour of Galois orbits of points of small height on a nondegenerate subvariey.

Condensed abstract: the ingredients for the proof of the uniform Mordell–Lang conjecture are recalled: they include the notion of nondegenerate subvariety, the height inequality and the equidistribution theorem for them.

Tips: the speaker should cover the content of [Gao21, sections 5,6] plus [Gao21, subsections 7.1 and 8.1]; there will little or no time for proofs. Recalling that this part of the seminar has more the spirit of a quick survey, the speaker is invited to make the nontrivial job of selecting only the most relevant parts; in particular, they could insist on [Gao21, Theorems 6.6, 7.1 and 8.1], and on the ideas around them.

11. A uniform Mordell–Lang for curves (04.06, *Roberto Gualdi*)

In this talk, the consequences of the results presented in the previous lecture are explored. First, the equidistribution theorem implies a *uniform Bogomolov-type statement*, giving a uniform bound for the number of points of small height on curves; this is reached by considering curves in family and by suitably adapting the classical Ullmo–Zhang's approach.

Considering the coarse moduli space and the universal curve of fixed genus over it, together with the corresponding universal Jacobian, the uniform Bogomolov-type statement and the height inequality are combined to obtain the *new gap principle*, which controls the number of small points on a curve.

Packing this result with the computation on the number of points of big height (independently achieved by de Diego and Rémond) finally yields the desired uniform Mordell–Lang statement.

Condensed abstract: the argument for the uniform Mordell–Lang for curves, which is based on a uniform Bogomolov-type result and on a new gap principle, is explained.

Tips: it is recommended to organize the talk around the three main results contained in it. First, it should be shown how the equidistribution theorem implies the uniform Bogomolov-type statement, following the detailed proof in [Gao21, subsection 8.2]. Secondly, the new gap principle from [Gao21, Theorem 4.1 and subsection 9.2] should be described, and some hint for the arguments can be given, taking the care to underline the role of the height inequality in the logic flow. Finally, the uniform Mordell–Lang should be deduced from the previous results as in [Gao21, subsection 9.3].

References

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