Geometrically simple counterexamples to a local-global principle for quadratic twists

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Local-global principles for isogenies

K a number field. Σ_K finite places of K.

$A, B/K$ abelian varieties of dimension $g \geq 1$.

$S \subseteq \Sigma_K$ primes of bad reduction of A and B.

$\forall \mathfrak{p} \in \Sigma_K \setminus S$, denote by $A_{K(\mathfrak{p})}, B_{K(\mathfrak{p})}$ the reductions of A, B modulo $\mathfrak{p}$.

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$\forall' \mathfrak{p} =$ For every prime ideal of $\mathcal{O}_K$ outside a 0 density set containing Σ.

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A and B are isogenous if and only if $A_{K(\mathfrak{p})}$ and $B_{K(\mathfrak{p})}$ are isogenous $\forall' \mathfrak{p}$.

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Theorem (Khare-Larsen; 2020)

$A_{\overline{K}}$ and $B_{\overline{K}}$ are isogenous if and only if $A_{\overline{K(\mathfrak{p})}}$ and $B_{\overline{K(\mathfrak{p})}}$ are isogenous $\forall' \mathfrak{p}$.

Twists

$F = K$, $K_{\mathfrak{p}}$, or $K(\mathfrak{p})$.

$A, B/F$ abelian varieties.

Category of abelian varieties up to isogeny:

- Objects: abelian varieties.
- Morphisms: $\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.

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We say that B is a twist of A if there exists an algebraic field extension L/F and an isogeny

$$\varphi : B_L \to A_L\,.$$

Quadratic twists

Let $G_F := \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of F.

$$\begin{array}{ccc} \{\text{Twists of } A \text{ (up to } F\text{-isogeny)}\} & \longrightarrow & H^1(G_F, \mathrm{Aut}^0(A_{\overline{F}})) \\ (\varphi : B_L \to A_L) & \mapsto & (\sigma \mapsto {}^\sigma\varphi \circ \varphi^{-1}) \end{array}$$

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The inclusion $\{\pm 1\} \subseteq \mathrm{Aut}^0(A_{\overline{F}})$ induces a (not necessarily injective) map

$$H^1(G_F, \{\pm 1\} = \mathrm{Hom}(G_F, \{\pm 1\}) \to H^1(G_F, \mathrm{Aut}^0(A_{\overline{F}}))$$

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B is called a quadratic twist of A if B is (isogenous to) A_χ for some χ.

Quadratic twists

Alternative more explicit description:

Given $\chi \in \mathrm{Hom}(G_F, \{\pm 1\})$, write $L := \overline{F}^{\ker(\chi)}$. Then

$$A_\chi = \begin{cases} \text{complement of } A \text{ in } \mathrm{Res}_{L/F}(A) & \text{if } \chi \text{ is nontrivial} \\ \text{A} & \text{if } \chi \text{ is trivial.} \end{cases}$$

Remark

Not every twist over a degree 2 extension L/K is a quadratic twist:

A^2 and $A \times A_\chi$ are twists over the a degree 2 extension,
but they are not quadratic twists.

Main result from Cardedeu 2021

K a number field and $A, B/K$ abelian varieties of dimension $g \geq 1$.

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Question Does the following hold?

A, B quadratic twists $\Leftrightarrow$ A, B locally quadratic twists. (LGP)

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$$A, B \text{ quadratic twists} \Leftrightarrow A, B \text{ locally quadratic twists.} \quad \text{(LGP)}$$

Theorem 1 (F.; 2021)

(LGP) holds for $g \leq 3$.
(LGP) does not hold for $g = 4$.

Remark

Ramakrishnan, Serre, and Rajan have given proofs of the above for $g = 1$.

Some ideas in the proof

Let ℓ be a prime. One translates the problem in terms of the ℓ-adic representation attached to A

$$\varrho_{A,\ell} : G_K \to \mathrm{Aut}(T_\ell(A) \otimes \mathbb{Q}_\ell), \qquad T_\ell(A) := \varprojlim_r A[\ell^r](\overline{K}) \simeq \mathbb{Z}_\ell^{2g}.$$

A and B are quadratic twists if and only if $\varrho_{A,\ell} \simeq \chi \otimes \varrho_{B,\ell}$ for χ quadratic.

Let K^{conn}/K be the minimal extension such that $\varrho_{A,\ell}(G_{K^{\mathrm{conn}}})^{\mathrm{Zar}}$ is connected.

Let K^{end}/K be the minimal extension such that $\mathrm{End}(A_{\overline{K}}) = \mathrm{End}(A_{K^{\mathrm{end}}})$.

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Proposition If A, B are locally quadratic twists, then:

- $\mathrm{End}(A_{\overline{K}}) \otimes \mathbb{Q} \simeq \mathrm{End}(B_{\overline{K}}) \otimes \mathbb{Q}$.
- $K^{\mathrm{conn}}(A) = K^{\mathrm{conn}}(B)$.
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Proposition If $K = K^{\mathrm{conn}}$ or $\mathrm{End}(A_{\overline{K}}) = \mathbb{Z}$, then (LGP) holds.

Some ideas in the proof II

Proof that (LGP) holds if $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \simeq \mathbb{Z}$

By Khare–Larsen, there is a finite extension L/K such that

$$\mathbb{Q}_\ell \simeq \mathrm{Hom}(A_L, B_L) \otimes \mathbb{Q}_\ell$$

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Note that

$$\mathrm{Hom}_{G_K}(\varrho_{B,\ell}, \chi \otimes \varrho_{A,\ell}) \simeq \mathrm{Hom}_{G_K}(\varrho_{A,\ell}^\vee \otimes \varrho_{B,\ell}, \chi) \neq 0\,.$$

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Proof that (LGP) holds if $K^{\mathrm{conn}} = K$

- By Serre and Tate, there exists $\Sigma \in \Sigma_K$ of density 1 such that

$$\mathrm{End}(A_{K(\mathfrak{p})}) \otimes \mathbb{Q} \simeq \mathrm{End}(A_{\overline{K(\mathfrak{p})}}) \otimes \mathbb{Q} \qquad \text{for every } \mathfrak{p} \in \Sigma.$$

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$$\begin{array}{ccc} H^1(G_K, E^\times/\{\pm 1\}) & \longrightarrow & \prod_{\mathfrak{p}\in\Sigma} H^1(D_\mathfrak{p}, E_\mathfrak{p}^\times/\{\pm 1\}) \\ & & \uparrow \\ & & \prod_{\mathfrak{p}\in\Sigma} H^1(G_{K(\mathfrak{p})}, (E_\mathfrak{p}^\times/\{\pm 1\})^{I(\mathfrak{p})}) \end{array}$$

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Remark

To complete the proof of Thm. 1, one goes through the possibilities for $\mathrm{End}(A_{\overline{K}}) \otimes \mathbb{Q}$ and K^{end}/K for $g \leq 3$.

Counterexample of dimension 6

- Let $\chi \neq \psi$ be quadratic characters and E an elliptic curve without CM.
- Observe the character table of the Klein group

	g_1	g_2	g_3	g_4
1	1	1	1	1
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- Then

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\begin{aligned}
1^{2\oplus} \oplus \chi^{2\oplus} \oplus \psi^{2\oplus}(g_1) &= 1^{3\oplus} \oplus \chi \oplus \psi \oplus \chi\psi(g_1) \\
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\end{aligned}$$

- Hence

$$E^2 \times E_\chi^2 \times E_\psi^2\,, \qquad E^3 \times E_\chi \times E_\psi \times E_{\chi\psi}$$

are locally quadratic twists, but not quadratic twists.

Kummer twists

A/K abelian variety such that $\mathbb{Q}(\zeta_{2m}) \subseteq \mathrm{End}(A_{\overline{K}}) \otimes \mathbb{Q} =: E$.

Let $\alpha \in K^\times$ and $m \in \mathbb{Z}_{\geq 1}$. The $2m$th Kummer twist of A by α, denoted A_α, is the one corresponding to the image of α under

$$K^\times/K^{\times,2m} \simeq H^1(G_K, \mu_{2m}) \to H^1(G_K, E^\times).$$

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Example

If A is the Jacobian of

$$C : y^2 = x^{m+1} + x/\mathbb{Q}\,,$$

then $(x, y) \mapsto (\zeta_m x, \zeta_{2m} y)$ induces $\mathbb{Q}(\zeta_{2m}) \subseteq E$. Observe that $\mathbb{Q}(\zeta_{\zeta_{2m}}) \subseteq \mathbb{Q}^{\mathrm{end}}$.

An easy caclulation shows that A_α is the Jacobian of

$$C_\alpha : y^2 = x^{m+1} + \alpha x/\mathbb{Q}\,.$$

(observe that $\phi_\alpha(x, y) = (\alpha^{1/m} x, \alpha^{(m+1)/(2m)} y)$ defines an isomorphism between C and C_α over $\overline{K}$).

Grunwald-Wang counterexamples

$A, B/K$ abelian varieties of dimension $g \geq 1$.

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Proposition (Ambrosi-Coppola-F.) Suppose that A and $\alpha \in K^\times$ are such that:

- $K^{\mathrm{end}} = K(\zeta_{2m})$ and $\mu_{2m} \cap K = \{\pm 1\}$.
- $\alpha, -\alpha \notin K^{\times, m}$.
- $\alpha \in K_\mathfrak{p}^{\times, m}$ $\forall' \mathfrak{p}$.

Then A, A_α are strongly locally quadratic twists, but not quadratic twists.

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Counterexample in dimension 4

Take $m = 8$, $\alpha = 16$, $K = \mathbb{Q}$. By Grunwald-Wang, the hypothesis of the Proposition are satisfied[1].

The Jacobians of $C : y^2 = x^9 + x/\mathbb{Q}$ and $C_{16} : y^2 = x^9 + 16x/\mathbb{Q}$ are strongly locally quadratic twists, but not quadratic twists.

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[1] Easier: For every odd p one of $\left(\frac{2}{p}\right)$, $\left(\frac{-2}{p}\right)$, $\left(\frac{-1}{p}\right)$ is 1 since the product of the three is 1.

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$\mathbb{Q}(\zeta_{16}) \subseteq \mathrm{End}(A_{\overline{\mathbb{Q}}}) \Rightarrow A_{\overline{K}}$ has CM.

By looking at the action of ζ_{16} on

$$\frac{dx}{y}, \quad \frac{xdx}{y}, \quad \frac{x^2dx}{y}, \quad \frac{x^3dx}{y},$$

one sees that $\mathrm{CM}(A) = \{1, 3, 5, 7\}$, under $\mathrm{Gal}(\mathbb{Q}(\zeta_{16})/\mathbb{Q}) \simeq (\mathbb{Z}/16\mathbb{Z})^\times$.

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Proposition If $K = K^{\mathrm{end}}$, then (LGP') holds.

Main result

Theorem 2 (Ambrosi-Coppola-F.)

Let $p \equiv 13 \pmod{24}$ be a prime.
Any abelian variety $A/\mathbb{Q}$ such that

$$\mathrm{End}(A_{\overline{\mathbb{Q}}}) = \mathbb{Q}(\zeta_{3p}) \qquad \text{and} \qquad \dim(A) = p - 1$$

has a twist B violating (LGP').

Remark

Such A's occur as quotients of Fermat curves (see Gallese-Goodson-Lombardo).

Cohomological translation

Let A/K be an abelian variety such that $E := \mathrm{End}(A_{\overline{K}})$ is a field.

Write $G := \mathrm{Gal}(K^{\mathrm{end}}/K)$.

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Proof:

$$\begin{array}{ccccccc} & & & H^1(G_K, \{\pm 1\}) & & \\ & & & \downarrow & & \\ & & & H^1(G_K, E^\times) & & \\ & & & \downarrow & & \\ 1 \longrightarrow & H^1(G, E^\times/\{\pm 1\}) & \longrightarrow & H^1(G_K, E^\times/\{\pm 1\}) & \longrightarrow & H^1(G_{K^{\mathrm{end}}}, E^\times/\{\pm 1\}) \\ & & & \downarrow & & \\ & & & H^2(G_K, \{\pm 1\}) & & \end{array}$$

Some ideas in the proof I

By Shimura, $E = \mathbb{Q}(\zeta_{3p}) = K^{\mathrm{end}}$. Hence

$$G := \mathrm{Gal}(E/\mathbb{Q}) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z}, \qquad G_1 := \mathrm{Gal}(E/\mathbb{Q}(\sqrt{-3})) = \mathbb{Z}/(p-1)\mathbb{Z}$$

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Cohomology for cyclic groups

Let $C = \langle g \rangle$ be a cyclic group and

$$1 \to N \to M \to Q \to 1$$

is an exact sequence of C-abelian groups, then

$$\begin{array}{ccc} H^1(C,Q) & \xrightarrow{\quad\delta\quad} & H^2(C,N) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Ker}(\mathcal{N}_C)/\langle g(q)q^{-1}\rangle_{q\in Q} & \xrightarrow{\mathcal{N}_C \circ \tilde{\cdot}} & N^C/\mathrm{Im}(\mathcal{N}_C) \end{array}$$

Some ideas in the proof II

Take $a, b \in \mathbb{Q}$ are such that $a^2 + 3b^2 = 3p$ (possible since $p \equiv 1 \pmod 3$).

Note that $y := \frac{a+b\sqrt{-3}}{\sqrt{-3p}} \in (E^\times/\{\pm 1\})^{G_1}$.

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Take $a, b \in \mathbb{Q}$ are such that $a^2 + 3b^2 = 3p$ (possible since $p \equiv 1 \pmod 3$).

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Note that $y \in \mathrm{Ker}(\mathcal{N}_{G/G_1})$. Hence it defines $y \in H^1(G/G_1, (E^\times/\{\pm 1\})^{G_1})$.

Defines $x = \mathrm{Inf}(y) \in H^1(G, E^\times/\{\pm 1\})$.

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Since $p \not\equiv 1 \pmod 8$ and $p \equiv 1 \pmod 4$, there exists $\mathbb{Q} \subseteq L \subseteq E = \mathbb{Q}(\zeta_{3p})$ such that $[L : \mathbb{Q}]$ is odd and $[E : L] = 8$. Then:

$$\text{(LGP') holds for } A \quad \Leftrightarrow \quad \text{(LGP') holds for } A_L.$$

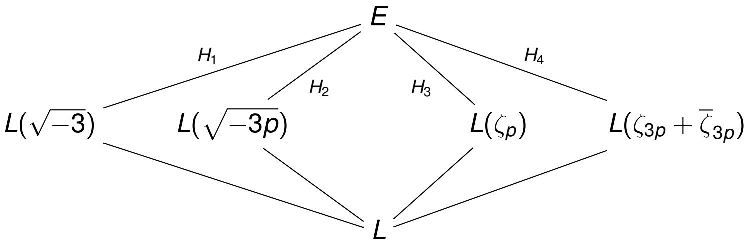
We may assume that $G = \mathrm{Gal}(E/L) \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

This has only 4 maximal cyclic groups $H_1 \simeq H_2 \simeq \mathbb{Z}/4\mathbb{Z}$, $H_3 \simeq H_4 \simeq \mathbb{Z}/2\mathbb{Z}$!

Some ideas in the proof III

Checking that x trivializes in $H^1(H_i, E^\times/\{\pm 1\})$ One checks that

$\delta : H^1(H_i, E^\times/\{\pm 1\}) \hookrightarrow H^2(H_i, \{\pm 1\})$ and computes $\delta(x) = 1$ via $\mathcal{N}_{H_i}$.



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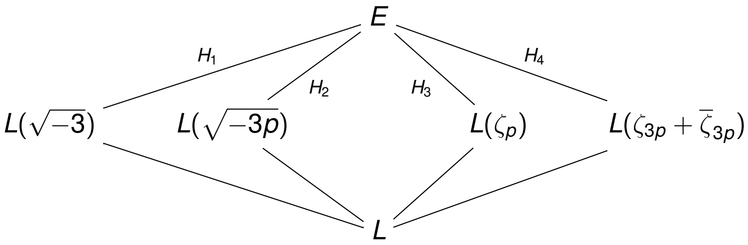

Checking that x trivializes in $H^2(G_{\mathbb{Q}}, \{\pm 1\})$

This can be checked locally thanks to the CFT s.e.s:

$$1 \to H^2(G_{\mathbb{Q}}, \{\pm 1\}) \to H^2(G_{\mathbb{R}}, \{\pm 1\}) \times \prod_{q \in \Sigma_{\mathbb{Q}}} H^2(G_{\mathbb{Q}_q}, \{\pm 1\}) \xrightarrow{\sum \mathrm{res}_q} \frac{1}{2}\mathbb{Z}/\mathbb{Z} \to 0.$$

For $q \neq 3, p$: $D_q \subseteq G$ is cyclic and the verification reduces to the above. In fact, D_p is also cyclic. The verification at $q = 3$ is automatic by CFT s.e.s.