Local-global principles for quadratic and polyquadratic twists of abelian varieties

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Local-global principles for isogenies of abelian varieties

K a number field.

$A, B/K$ abelian varieties of dimension $g \geq 1$.

Σ the set of primes of bad reduction of A and B.

$\forall \mathfrak{p} \notin \Sigma$, denote by $A_\mathfrak{p}, B_\mathfrak{p}/K(\mathfrak{p})$ the reductions of A, B modulo $\mathfrak{p}$.

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$\forall' \mathfrak{p} =$ For every prime ideal of $\mathcal{O}_K$ outside a 0 density set containing Σ.

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Theorem (Khare-Larsen; 2020)

$\overline{A}$ and $\overline{B}$ are isogenous if and only if $\overline{A}_\mathfrak{p}$ and $\overline{B}_\mathfrak{p}$ are isogenous $\forall' \mathfrak{p}$.
Here $\overline{A} := A \times_K \overline{K}$, $\overline{A}_\mathfrak{p} := A_\mathfrak{p} \times_{K(\mathfrak{p})} \overline{K(\mathfrak{p})}$.

Polyquadratic twists

$F = K$ or $K(\mathfrak{p})$.

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Category of abelian varieties up to isogeny:

- Objects: abelian varieties.
- Morphisms: $\mathrm{Hom}^0(A, B) := \mathrm{Hom}(A, B) \otimes_{\mathbb{Z}} \mathbb{Q}$.

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We say that B is a twist of A if there exists an algebraic field extension L/F and an isogeny

$$\varphi : B_L \to A_L \, .$$

Here $A_L := A \times_F L$, $B_L := B \times_F L$.

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Here $A_L := A \times_F L$, $B_L := B \times_F L$.

We say that A and B are polyquadratic twists (of degree 2^r) if they become isogenous over the compositum of r quadratic extensions of F.

Quadratic twists

Let $G_F := \mathrm{Gal}(\overline{F}/F)$ be the absolute Galois group of F.

$$\begin{array}{ccc} \{\text{Twists of } A \text{ (up to } F\text{-isogeny)}\} & \longrightarrow & H^1(G_F, \mathrm{Aut}^0(A_{\overline{F}})) \\ (\varphi : B_L \to A_L) & \mapsto & (\sigma \mapsto {}^{\sigma}\varphi \circ \varphi^{-1}) \end{array}$$

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Weil descent

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For $\chi \in H^1(G_F, \{\pm 1\}) = \mathrm{Hom}(G_F, \{\pm 1\})$, let A_χ be the twist of A assoc. to χ.

We say that B is a quadratic twist of A if B is (isogenous to) A_χ for some χ.

Quadratic twists

Alternative more explicit description:

Write $L := \overline{F}^{\ker(\chi)}$

$$A_\chi = \begin{cases} \text{complement of } A \text{ in } \mathrm{Res}_{L/F}(A) & \text{if } \chi \text{ is nontrivial} \\ \text{A} & \text{if } \chi \text{ is trivial.} \end{cases}$$

Remark

Not every polyquadratic twist of degree 2 is a quadratic twist.

Example

A^2 and $A \times A_\chi$ are polyquadratic twists of degree 2, but in general they will not be quadratic twists.

Main results

K a number field and $A, B/K$ abelian varieties of dimension $g \geq 1$.

Theorem 1 (F.; 2021)

For $g \leq 3$:
A, B are quadratic twists if and only if $A_\mathfrak{p}, B_\mathfrak{p}$ are quadratic twists $\forall' \mathfrak{p}$.
Moreover the above is false for $g = 4$.

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Theorem 2 (F.-Perucca; 2022)

For $g \leq 2$:
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Moreover the above is false for $g \geq 3$.

Remark

Ramakrishnan, Serre, and Rajan have given proofs of the above for $g = 1$.

Question

For which dimensions can one extend Theorem 1?

Counterexamples

Counterexample of dimension 4 (E. Costa). The Jacobians of

$$y^2 = x^9 + x/\mathbb{Q}\,, \qquad y^2 = x^9 + 16x/\mathbb{Q}$$

are locally quadratic twists at all odd primes, but they are not quadratic twists.

Costa has found similar counterexamples within the family $y^2 = x^{2g+1} + ax$ for

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Counterexample of dimension 6. Let χ, ψ be distinct quadratic characters and E an elliptic curve without CM. An easy exercise using the character table of the Klein group shows that

$$E^2 \times E_\chi^2 \times E_\psi^2\,, \qquad E^3 \times E_\chi \times E_\psi \times E_{\chi\psi}$$

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Counterexample of dimension 5?

Representation theoretic setting

E a topological field.

G a compact topological group.

$\varrho, \varrho' : G \to \mathrm{GL}_r(E)$ semisimple continuous representations.

We say that ϱ and ϱ' are quadratic twists if $\varrho' \simeq \chi \otimes \varrho$ holds for some $\chi \in \mathrm{Hom}(G, \{\pm 1\})$.

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We say that ϱ and ϱ' are polyquadratic twists if

$$\varrho \simeq \bigoplus_{i=1}^{t} \varrho_i \qquad \text{and} \qquad \varrho' \simeq \bigoplus_{i=1}^{t} \varrho'_i\,,$$

where $\varrho_i, \varrho'_i : G \to \mathrm{GL}_{r_i}(E)$ are quadratic twists for all i.

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Proposition

ϱ, ϱ' are polyquadratic twists if and only if $\varrho|_H \simeq \varrho'|_H$ for some $H \trianglelefteq G$ such that G/H is a finite abelian group of exponent dividing 2.

Example

K a number field.

$A, B/K$ abelian varieties of dimension $g \geq 1$.

ℓ a prime. The ℓ-adic Tate module of A is

$$T_\ell(A) := \varprojlim_r A[\ell^r](\overline{K}) \simeq \mathbb{Z}_\ell^{2g}\,, \qquad V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell\,.$$

We denote by $\varrho_{A,\ell}$ the representation of G_K afforded by $V_\ell(A)$.

Faltings isogeny thm. (as on slide 1) with the Brauer-Nesbitt thm. imply:

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Proposition

A, B are isogenous if and only if $\varrho_{A,\ell} \simeq \varrho_{B,\ell}$.

A, B are quadratic twists if and only if $\varrho_{A,\ell}\,,\, \varrho_{B,\ell}$ are quadratic twists.

A, B are polyquadratic twists if and only if $\varrho_{A,\ell}\,,\, \varrho_{B,\ell}$ are polyquadratic twists.

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We say that ϱ and ϱ' are locally quadratic twists if any of the following equivalent conditions hold:

- For every $s \in G$, $\det(1 - \varrho(s)T) = \det(1 - \epsilon_s \varrho'(s)T)$ for some $\epsilon_s \in \{\pm 1\}$.
- $\varrho^{\otimes 2} \simeq \varrho'^{\otimes 2}\,, \quad \wedge^{2i}\varrho \simeq \wedge^{2i}\varrho'\,, \quad \varrho \otimes \wedge^{2i+1}\varrho \simeq \varrho' \otimes \wedge^{2i+1}\varrho'$ for all i.

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Remark

$A_{\mathfrak{p}}, B_{\mathfrak{p}}$ (poly)quadratic twists $\forall' \mathfrak{p} \iff \varrho_{A,\ell}, \varrho_{B,\ell}$ locally (poly)quadratic twists.

Question

Are the below implications in fact equivalences?

1) ϱ and ϱ' quadratic twists $\Longrightarrow$ ϱ and ϱ' locally quadratic twists.

2) ϱ and ϱ' polyquadratic twists $\Longrightarrow$ ϱ and ϱ' locally polyquadratic twists.

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The converse implication is true if $\deg(\varrho) = 2$ (Ramakrishnan) or odd;
False for $\deg(\varrho) = 4$ (Chidambaran) or 6.

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Corollary

Suppose that $g = 1$.

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Remark

The ϱ, ϱ' in the above counterexamples in degrees 4 and 6 do *not* correspond to ℓ-adic representations of abelian surfaces or threefolds.

Rajan's theorem (Locally quadratic twist $\Rightarrow$ Twist)

Theorem (Rajan)

Let $\varrho, \varrho : G_K \to \mathrm{GL}_r(\mathbb{Q}_\ell)$ be semisimple. Suppose:

- $\varrho(G_K)^{\mathrm{Zar}}$ is connected.
- $\mathrm{Dens}\big(\{\mathfrak{p} \mid \mathrm{Tr}(\varrho(\mathrm{Frob}_\mathfrak{p})) = \mathrm{Tr}(\varrho'(\mathrm{Frob}_\mathfrak{p}))\}\big) > 0$.

Then there exists a finite L/K such that $\varrho|_{G_L} \simeq \varrho'|_{G_L}$.

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Corollary

If A and B are locally quadratic twists, then A and B are twists.
In particular, $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \simeq \mathrm{End}(B_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$.

Faltings Isogeny theorem (Loc. quad. twist $\Rightarrow$ same end. field)

Let K_A denote the minimal extension such that $\mathrm{End}(A_{K_A}) = \mathrm{End}(A_{\overline{\mathbb{Q}}})$.

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If A and B are locally quadratic twists, then $K_A = K_B$.

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Indeed, if A and B are locally quadratic twists, then

$$(\mathrm{Tr}\,\varrho_A(\mathrm{Frob}_\mathfrak{p}))^2 = (\mathrm{Tr}\,\varrho_B(\mathrm{Frob}_\mathfrak{p}))^2 \qquad \forall' \mathfrak{p}\,.$$

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$$\varrho_A \otimes \varrho_A \simeq \varrho_B \otimes \varrho_B$$

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Then for any extension L/K

$$\mathrm{End}(A_L) \otimes \mathbb{Q}_\ell \simeq (\varrho_A \otimes \varrho_A^\vee)^{G_L} \simeq (\varrho_B \otimes \varrho_B^\vee)^{G_L} \simeq \mathrm{End}(B_L) \otimes \mathbb{Q}_\ell\,.$$

The case $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \simeq \mathbb{Z}$

The proof of Theorem 1 is by cases on the possibilities for $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$.

The case $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \simeq \mathbb{Z}$

By Rajan's Theorem, there is a finite extension L/K such that

$$\mathbb{Q}_\ell \simeq \mathrm{Hom}(A_L, B_L) \otimes \mathbb{Q}_\ell \simeq \mathrm{Hom}_{G_L}(\varrho_{A,\ell}, \varrho_{B,\ell}) \simeq (\varrho_{A,\ell}^\vee \otimes \varrho_{B,\ell})^{G_L}\,.$$

$(\varrho_{A,\ell}^\vee \otimes \varrho_{B,\ell})^{G_L}$ affords a character χ of $\mathrm{Gal}(L/K)$, which in fact is quadratic.

It will suffice to see that

$$\mathrm{Hom}_{G_K}(\varrho_{B,\ell}, \chi \otimes \varrho_{A,\ell}) \neq 0\,.$$

Note that

$$\mathrm{Hom}_{G_K}(\varrho_{B,\ell}, \chi \otimes \varrho_{A,\ell}) \simeq \mathrm{Hom}_{G_K}(\varrho_{A,\ell}^\vee \otimes \varrho_{B,\ell}, \chi) \neq 0\,.$$

The case $A_{\overline{\mathbb{Q}}} \sim E^2$

Suppose we are in the almost antagonic case:

$A_{\overline{\mathbb{Q}}} \sim E^2$, where $E/\overline{\mathbb{Q}}$ is an elliptic curve without CM.

Theorem (F.-Guitart)

There exists a finite Galois extension L/K, a number field M, and

- An Artin representation $\theta : \mathrm{Gal}(L/K) \to \mathrm{GL}_2(M)$.
- For every ℓ totally split in M, a strongly absolutely irreducible M-rational ℓ-adic representation $\varrho : G_K \to \mathrm{GL}_2(\mathbb{Q}_\ell)$

such that $\varrho_{A,\ell} \simeq \theta \otimes_{\mathbb{Q}_\ell} \varrho$.

Using the previous theorem, we can write

$$\varrho_{A,\ell} \simeq \theta \otimes \varrho\,, \qquad \varrho_{B,\ell} \simeq \theta' \otimes \varrho'\,.$$

After enlarging L/K, we can assume that

$$\theta, \theta' : \mathrm{Gal}(L/K) \to \mathrm{GL}_2(M) \quad \text{and} \quad \varrho_{A,\ell}|_{G_L} \simeq \varrho_{B,\ell}|_{G_L}\,.$$

In particular $\varrho|_{G_L} \simeq \varrho'|_{G_L}$. Hence there exists a character χ of $\mathrm{Gal}(L/K)$ such that $\varrho' \simeq \chi \otimes \varrho$.

That A and B are locally quadratic twists means that

$$\det(1 - \theta' \otimes \chi \otimes \varrho(\mathrm{Frob}_\mathfrak{p})T) = \det(1 \pm \theta \otimes \varrho(\mathrm{Frob}_\mathfrak{p})T) \qquad \forall' \mathfrak{p}\,.$$

Let $\alpha_{1,\mathfrak{p}}, \alpha_{2,\mathfrak{p}}$ be the eigenvalues of $\varrho(\mathrm{Frob}_\mathfrak{p})$. One can show that

$$\frac{\alpha_{1,\mathfrak{p}}}{\alpha_{2,\mathfrak{p}}} \notin \mu_\infty \qquad \forall' \mathfrak{p}\,.$$

$$\prod_{i=1}^{2} \det(1 - \theta' \otimes \chi(\mathrm{Frob}_{\mathfrak{p}})\alpha_{i,\mathfrak{p}} T) = \prod_{i=1}^{2} \det(1 \pm \theta(\mathrm{Frob}_{\mathfrak{p}})\alpha_{i,\mathfrak{p}} T) \qquad \forall' \mathfrak{p}\,.$$

One deduces

$$\det(1 - \theta' \otimes \chi(\mathrm{Frob}_{\mathfrak{p}}) T) = \det(1 \pm \theta(\mathrm{Frob}_{\mathfrak{p}}) T) \qquad \forall' \mathfrak{p}\,.$$

By Chebotarev, this means that $\theta' \otimes \chi$ and θ are locally quadratic twists.

By Ramakrishnan's theorem, there exists a quadratic character ψ such that

$$\theta' \otimes \chi \simeq \psi \otimes \theta\,.$$

Hence

$$\varrho_{B,\ell} \simeq \theta' \otimes \chi \otimes \varrho \simeq \psi \otimes \theta \otimes \varrho \simeq \psi \otimes \varrho_{A,\ell}\,.$$

Gràcies per la vostra atenció!

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Danke für Ihre Aufmerksamkeit!

Merci pour votre attention!

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