Sato–Tate groups of abelian threefolds

Francesc Fité (MIT), Kiran S. Kedlaya (UCSD), A.V. Sutherland (MIT)

Harvard Number Theory Seminar.
25th March 2020.

A preview of the classification: https://arxiv.org/abs/1911.02071

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2. Sato–Tate groups of abelian varieties of dimension ≤ 3
3. Statement of the main results
4. Abelian threefolds: The classification problem
5. Abelian threefolds: The realization problem

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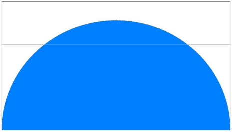
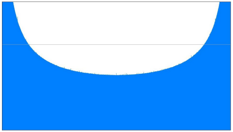
Sato–Tate groups of elliptic curves

- k a number field.
- E/k an elliptic curve.
- The Sato–Tate group $\mathrm{ST}(E)$ is defined as:
  - $\mathrm{SU}(2)$ if E does not have CM.
  - $\mathrm{U}(1) = \left\{ \begin{pmatrix} u & 0 \\ 0 & \overline{u} \end{pmatrix} : u \in \mathbb{C}, |u| = 1 \right\}$ if E has CM by $M \subseteq k$.
  - $N_{\mathrm{SU}(2)}(\mathrm{U}(1))$ if E has CM by $M \not\subseteq k$.
- Note that $\mathrm{Tr}\colon \mathrm{ST}(E) \to [-2, 2]$. Denote $\mu = \mathrm{Tr}_*(\mu_{\mathrm{Haar}})$.

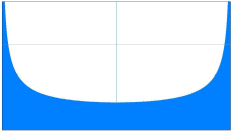
$\mathrm{SU}(2)$



$\mathrm{U}(1)$



$N(\mathrm{U}(1))$



The Sato–Tate conjecture for elliptic curves

- For a prime $\mathfrak{p}$ of good reduction for E, set

$$a_{\mathfrak{p}} := N(\mathfrak{p}) + 1 - \#E(\mathbb{F}_{\mathfrak{p}}) = \mathrm{Tr}(\mathrm{Frob}_{\mathfrak{p}} \,|\, V_\ell(E)) . \qquad (\text{for } \mathfrak{p} \nmid \ell)$$

- The normalized Frobenius trace satisfies

$$\overline{a}_{\mathfrak{p}} := \frac{a_{\mathfrak{p}}}{\sqrt{N(\mathfrak{p})}} \in [-2, 2] .$$

Sato–Tate conjecture

The sequence $\{\overline{a}_{\mathfrak{p}}\}_{\mathfrak{p}}$ is equidistributed on $[-2, 2]$ w.r.t μ.

- If $\mathrm{ST}(E) = \mathrm{U}(1)$ or $N(\mathrm{U}(1))$: Known in full generality (Hecke, Deuring).
- Known if $\mathrm{ST}(E) = \mathrm{SU}(2)$ and k is totally real. (Barnet-Lamb, Clozel, Gee, Geraghty, Harris, Shepherd-Barron, Taylor);
- Known if $\mathrm{ST}(E) = \mathrm{SU}(2)$ and k is a CM field (Allen,Calegari,Caraiani,Gee,Helm,LeHung,Newton,Scholze,Taylor,Thorne).

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Toward the Sato–Tate group: the ℓ-adic image

- Let A/k be an abelian variety of dimension $g \geq 1$.
- Consider the ℓ-adic representation attached to A

$$\varrho_{A,\ell}\colon G_k \to \mathrm{Aut}(V_\ell(A))\,.$$

- Serre defines $\mathrm{ST}(A)$ in terms of $\mathcal{G}_\ell = \varrho_{A,\ell}(G_k)^{\mathrm{Zar}} \subseteq \mathrm{GSp}_{2g}/\mathbb{Q}_\ell$.
- For $g \leq 3$, Banaszak and Kedlaya describe $\mathrm{ST}(A)$ *in terms of endomorphisms*.
- Recall there is a G_k-equivariant monomorphism

$$\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{End}_{\mathcal{G}_\ell^0}(V_\ell(A))$$

  (by Faltings, in fact an isomorphism).
- More conveniently

$$\mathcal{G}_\ell^0 \hookrightarrow \{\gamma \in \mathrm{GSp}_{2g}/\mathbb{Q}_\ell \,|\, \gamma\alpha\gamma^{-1} = \alpha \text{ for all } \alpha \in \mathrm{End}(A_{\overline{\mathbb{Q}}})\}\,.$$

The twisted Lefschetz group

- More accurately

$$\mathcal{G}_\ell \hookrightarrow \bigcup_{\sigma \in G_k} \{\gamma \in \mathrm{GSp}_{2g}/\mathbb{Q}_\ell \,|\, \gamma\alpha\gamma^{-1} = \sigma(\alpha) \text{ for all } \alpha \in \mathrm{End}(A_{\overline{\mathbb{Q}}})\}\,.$$

- For $g = 4$, Mumford has constructed A/k such that

$$\mathrm{End}(A_{\overline{\mathbb{Q}}}) \simeq \mathbb{Z} \quad \text{and} \quad \mathcal{G}_\ell \subsetneq \mathrm{GSp}_{2g}(\mathbb{Q}_\ell)\,.$$

- For $g \leq 3$, one has

$$\mathcal{G}_\ell \simeq \bigcup_{\sigma \in G_k} \{\gamma \in \mathrm{GSp}_{2g}/\mathbb{Q}_\ell \,|\, \gamma\alpha\gamma^{-1} = \sigma(\alpha) \text{ for all } \alpha \in \mathrm{End}(A_{\overline{\mathbb{Q}}})\}\,.$$

Definition

The Twisted Lefschetz group is defined as

$$\mathrm{TL}(A) = \bigcup_{\sigma \in G_k} \{\gamma \in \mathrm{Sp}_{2g}/\mathbb{Q} | \gamma\alpha\gamma^{-1} = \sigma(\alpha) \text{ for all } \alpha \in \mathrm{End}(A_{\overline{\mathbb{Q}}})\}.$$

The Sato–Tate group when $g \leq 3$

- From now on, assume $g \leq 3$.

Definition

$\mathrm{ST}(A) \subseteq \mathrm{USp}(2g)$ is a maximal compact subgroup of $\mathrm{TL}(A)(\mathbb{C})$.

- Note that

$$\mathrm{ST}(A)/\,\mathrm{ST}(A)^0 \simeq \mathrm{TL}(A)/\,\mathrm{TL}(A)^0 \simeq \mathrm{Gal}(F/k)\,.$$

  where F/k is the minimal extension such that $\mathrm{End}(A_F) \simeq \mathrm{End}(A_{\overline{\mathbb{Q}}})$. We call F the endomorphism field of A.
- To each prime $\mathfrak{p}$ of good reduction for A, one can attach an element

$$x_{\mathfrak{p}} = \text{``}\mathrm{Conj}\left(\frac{\varrho_{A,\ell}(\mathrm{Frob}_{\mathfrak{p}})}{\sqrt{N(\mathfrak{p})}}\right)\text{''} \in \mathrm{Conj}(\mathrm{ST}(A))\,.$$

Sato–Tate conjecture for abelian varieties

The sequence $\{x_{\mathfrak{p}}\}_{\mathfrak{p}}$ is equidistributed on $\mathrm{Conj}(\mathrm{ST}(A))$ w.r.t the push forward of the Haar measure of $\mathrm{ST}(A)$.

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Sato–Tate axioms for $g \leq 3$

The Sato–Tate axioms for a closed subgroup $G \subseteq \mathrm{USp}(2g)$ for $g \leq 3$ are:

Hodge condition (ST1)

There is a homomorphism $\theta\colon \mathrm{U}(1) \to G^0$ such that $\theta(u)$ has eigenvalues u and $\overline{u}$ each with multiplicity g. The image of such a θ is called a *Hodge circle*. Moreover, the Hodge circles generate a dense subgroup of G^0.

Rationality condition (ST2)

For every connected component $H \subseteq G$ and for every irreducible character $\chi\colon \mathrm{GL}_{2g}(\mathbb{C}) \to \mathbb{C}$:

$$\int_H \chi(h)\mu_{\mathrm{Haar}} \in \mathbb{Z}\,,$$

where μ_{Haar} is normalized so that $\mu_{\mathrm{Haar}}(G^0) = 1$.

Lefschetz condition (ST3)

$$\{\gamma \in \mathrm{USp}(2g) | \gamma\alpha\gamma^{-1} = \alpha \text{ for all } \alpha \in \mathrm{End}_{G^0}(\mathbb{C}^{2g})\} = G^0\,.$$

General remarks and dimension $g = 1$

Proposition

If $G = \mathrm{ST}(A)$ for some A/k with $g \leq 3$, then G satisfies the ST axioms.

$$\begin{array}{rcl} \text{Mumford–Tate conjecture} & \rightsquigarrow & \text{(ST1)} \\ \text{“Rationality” of } \mathcal{G}_\ell & \rightsquigarrow & \text{(ST2)} \\ \text{Bicommutant property of } \mathcal{G}_\ell^0 & \rightsquigarrow & \text{(ST3)} \end{array}$$

- Axioms (ST1), (ST2) are expected for general g. But not (ST3)!

Remark ($g = 1$)

- Up to conjugacy, 3 subgroups of $\mathrm{USp}(2)$ satisfy the ST axioms.
- All 3 occur as ST groups of elliptic curves defined over number fields.
- Only 2 of them occur as ST groups of elliptic curves defined over $\mathbb{Q}$.

Sato–Tate groups for $g = 2$

Theorem (F.-Kedlaya-Rotger-Sutherland; 2012)

- Up to conjugacy, 55 subgroups of USp(4) satisfy the ST axioms.
- 52 of them occur as ST groups of abelian surfaces over number fields.
- 34 of them occur as ST groups of abelian surfaces over $\mathbb{Q}$.

Corollary

The degree of the endomorphism field of an abelian surface over a number field divides 48.

(this refines previous results by Silverberg).

Theorem (Johansson, N. Taylor; 2014-19)

For $g = 2$ and $k = \mathbb{Q}$, the ST conjecture holds for 33 of the 34 possible ST groups.

Sato–Tate groups for $g = 3$

Theorem(F.-Kedlaya-Sutherland; 2019)

- Up to conjugacy, 433 subgroups of USp(6) satisfy the ST axioms.
- Only 410 of them occur as Sato–Tate groups of abelian threefolds over number fields.

Corollary

The degree of the endomorphism field $[F : \mathbb{Q}]$ of an abelian threefold over a number field divides 192, 336, or 432.

- This refines a previous result of Guralnick and Kedlaya, which asserts

$$[F : \mathbb{Q}] \mid 2^6 \cdot 3^3 \cdot 7 = \mathrm{Lcm}(192, 336, 432)\,.$$

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Classification: identity components

(ST1) and (ST3) allow 14 possibilities for $G^0 \subseteq \mathrm{USp}(6)$:

- $\mathrm{USp}(6)$
- $\mathrm{U}(3)$
- $\mathrm{SU}(2) \times \mathrm{USp}(4)$
- $\mathrm{U}(1) \times \mathrm{USp}(4)$
- $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
- $\mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$
- $\mathrm{SU}(2) \times \mathrm{SU}(2)_2$
- $\mathrm{SU}(2) \times \mathrm{U}(1)_2$
- $\mathrm{U}(1) \times \mathrm{SU}(2)_2$
- $\mathrm{U}(1) \times \mathrm{U}(1)_2$
- $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$
- $\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$
- $\mathrm{SU}(2)_3$
- $\mathrm{U}(1)_3$

Notations:

- For $d \in \{1,3\}$:

$$\mathrm{U}(d) = \begin{pmatrix} \mathrm{U}(d)^{\mathrm{St}} & 0 \\ 0 & \overline{\mathrm{U}(d)}^{\mathrm{St}} \end{pmatrix} \subseteq \mathrm{USp}(2d)$$

- For $d \in \{2,3\}$ and $H \in \{\mathrm{SU}(2), \mathrm{U}(1)\}$:

$$H_d = \{\mathrm{diag}(u, \overset{d}{\ldots}, u) \,|\, u \in H\}$$

- Note in particular that

$$\mathrm{SU}(2) \times \mathrm{U}(1)_2 \simeq \mathrm{U}(1) \times \mathrm{SU}(2)_2 \,.$$

Determining the possibilities for G for fixed G^0

- Compute $N = N_{\mathrm{USp}(6)}(G^0)$ and ${}^{N}/_{G^0}$.
- Use

$$\left\{ \begin{array}{c} \mathcal{G} \subseteq \mathrm{USp}(6) \text{ with } \mathcal{G}^0 = G^0 \\ \text{satisfying (ST2)} \end{array} \right\} /\sim \;\longleftrightarrow\; \left\{ \begin{array}{c} \text{finite } H \subseteq N/G^0 \text{ s.t.} \\ HG^0 \text{ satisfies (ST2)} \end{array} \right\} /\sim$$

- Consider 3 cases:
  - **Genuine of dimension** 3: $G^0 \subseteq \mathrm{USp}(6)$ cannot be written as

$$G^0 = G^{0,1} \times G^{0,2} \text{ with } G^{0,1} \subseteq \mathrm{SU}(2) \text{ and } G^{0,2} \subseteq \mathrm{USp}(4)\,. \qquad (*)$$

  - **Split case**: G^0 can be written as in (*) and

$$N \simeq N_1 \times N_2\,, \qquad \text{where } N_i = N_{\mathrm{USp}(2i)}(G^{0,i})\,.$$

  - **Non-split case**: G^0 can be written as in (*) and

$$N_1 \times N_2 \subsetneq N\,.$$

Classification: cases depending on G^0

$$
\text{Genuine dim. 3 cases} \begin{cases} \mathrm{USp}(6) \\ \mathrm{U}(3) \end{cases}
$$

$$
\text{Split cases} \begin{cases} \mathrm{SU}(2) \times \mathrm{USp}(4) \\ \mathrm{U}(1) \times \mathrm{USp}(4) \\ \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \\ \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1) \\ \mathrm{SU}(2) \times \mathrm{SU}(2)_2 \\ \mathrm{SU}(2) \times \mathrm{U}(1)_2 \\ \mathrm{U}(1) \times \mathrm{SU}(2)_2 \\ \mathrm{U}(1) \times \mathrm{U}(1)_2 \end{cases}
$$

$$
\text{Non-split cases} \begin{cases} \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \\ \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1) \\ \mathrm{SU}(2)_3 \\ \mathrm{U}(1)_3 \end{cases}
$$

Classification: From G^0 to G

- Genuine cases: USp(6), U(3), N(U(3)).
- Split cases. The determination of

$$\mathcal{A} = \left\{ \begin{array}{c} H \subseteq N/G^0 \text{ finite s.t.} \\ HG^0 \text{ satisfies (ST2)} \end{array} \right\} /\sim$$

  is facilitated by fact that $N \simeq N_1 \times N_2$: H must be a fiber product of finite groups encountered in the classifications in dimensions 1 and 2. This accounts for 211 groups.
- Non-split cases:

G^0	N/G^0	$\#\mathcal{A}$
$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	S_3	4
$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$	$(C_2 \times C_2 \times C_2) \rtimes S_3$	33
$\mathrm{SU}(2)_3$	$\mathrm{SO}(3)$	11
$\mathrm{U}(1)_3$	$\mathrm{PSU}(3) \rtimes C_2$	171

$G^0 = \mathrm{U}(1)_3$: Ingredients of the proof

- The finite $\mu_3 \subseteq H \subseteq \mathrm{SU}(3)$ were classified by Blichfeldt, Miller, and Dickson (1916). They are:
  - Abelian groups
  - C_2-extensions of abelian groups.
  - C_3-extenions of abelian groups.
  - S_3-extensions of abelian groups.
  - cyclic extensions of exceptional subgroups of SU(2) ($2T$, $2O$, $2I$).
  - Exceptional subgroups of SU(3)
    (projected in PSU(3) are $E(36)\,, E(72)\,, E(216)\,, A_5,\ A_6,\ E(168)$).
- Determining the possible orders of $h \in H$:
  - (ST2) implies that $|\operatorname{Tr}(h)|^2 \in \mathbb{Z}$.
  - If $z_1, z_2, z_3 \in \mu_\infty$ are the eigenvalues of h, then:

    $$|z_1 + z_2 + z_3|^2 \in \mathbb{Z} \text{ and } z_1 z_2 z_3 = 1\,.$$

  - One deduces that $\mathrm{ord}(h) | 21, 24, 36$.
- Assemble elements to build groups of the shape described by the BMD classification.
- Build C_2-extensions of H.

Classification: Invariants

- Only 210 distinct pairs $(G^0, G/G^0)$.
- Define the (i, j, k)-**th moment**, for $i, j, k \geq 0$, as

$$\mathrm{M}_{i,j,k}(G) := \dim_{\mathbb{C}} \left((\wedge^1 \mathbb{C}^6)^{\otimes i} \otimes (\wedge^2 \mathbb{C}^6)^{\otimes j} \otimes (\wedge^3 \mathbb{C}^6)^{\otimes k} \right)^G \in \mathbb{Z}_{\geq 0} .$$

- The sequence $\{\mathrm{M}_{i,j,k}(G)\}_{i,j,k}$ attains 432 values. It only conflates a pair of groups G_1, G_2, for which however

$$G_1/G_1^0 \simeq \langle 54, 5 \rangle \not\simeq \langle 54, 8 \rangle \simeq G_2/G_2^0 .$$

- In total, the 433 groups have 10988 connected components (4 for $g = 1$ and 414 for $g = 2$).
- Any possible order of G/G^0 divides 192, 336, or 432.

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Realization: upper bound

- For $G^0 = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$, we have found 33 subgroups in $N/G^0 \simeq (C_2 \times C_2 \times C_2) \rtimes S_3$ (all satisfying (ST2)).
- However, if G is realizable by A, then
  - A is isogenous to a product of abelian varieties A_i with CM by M_i.
  - $G/G^0 \simeq \mathrm{Gal}(F/k) \simeq \prod \mathrm{Gal}(kM_i^*/k) \subseteq C_2 \times C_2 \times C_2, C_2 \times C_4, C_6$.
- This rules out 20 of the 33 subgroups of N/G^0.
- For $G^0 = \mathrm{SU}(2) \times \mathrm{U}(1) \times \mathrm{U}(1)$, a similar logic rules out 3 of the subgroups in $N/G^0 \simeq D_4$ (all satisfying (ST2)). These correspond to the 3 subgroups which satisfy the ST axioms in dimension 2, but do not arise as ST groups.
- This leaves 433-20-3=410 groups.
- It suffices to realize the 33 maximal groups (for prescribed identity component). Finite index subgroups are realized by base change.

Realization of the maximal groups

- Genuine cases (2 max. groups):
  - $\mathrm{USp}(6)$: generic case. Eg.: $y^2 = x^7 - x + 1/\mathbb{Q}$.
  - $N(\mathrm{U}(3))$: Picard curves. Eg.: $y^3 = x^4 + x + 1/\mathbb{Q}$.
- Split cases (13 max. groups):
  Maximality ensures the triviality of the fiber product, i.e.

$$G \simeq G_1 \times G_2 \,,$$

  where G_1 and G_2 are realizable in dimensions 1 and 2.
- Triple products (4 max. groups):
  - $G^0 = \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ (1. max. group): $\mathrm{Res}^L_{\mathbb{Q}}(E)$, where $L/\mathbb{Q}$ a non-normal cubic and E/L e.c. which is not a $\mathbb{Q}$-curve.
  - $G^0 = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ (3 max. groups): Products of CM abelian varieties.

Realization of the maximal groups

- $G^0 = \mathrm{SU}(2)_3$ (2 max. groups: S_4, D_6): Twists of cubes of non CM elliptic curves.
  - Take a non CM elliptic curve E.
  - Consider a faithful representation

$$\xi : \mathrm{Gal}(L/\mathbb{Q}) \simeq S_4 \to \mathrm{GL}_3(\mathbb{Z})\,.$$

  - Let $A = E^3$ and $A_{\tilde{\xi}}$ be the twist of A by

$$\tilde{\xi} : \mathrm{Gal}(L/\mathbb{Q}) \simeq S_4 \to \mathrm{Aut}(A)\,.$$

- $G^0 = \mathrm{U}(1)_3$ (12 max. groups): Twists of cubes of CM elliptic curves.