On the convergence rate towards the Sato-Tate measure

Francesc Fité (UPC) and Xavier Guitart (UB)

Lleida, 28th June 2017.

Generalized Sato–Tate conjecture

- k is a number field.
- A/k is an abelian variety of dimension $g \geq 1$.
- $\varrho_A \colon G_k \to \mathrm{Aut}(V_\ell(A))$ is the ℓ-adic representation attached to A.
- For each prime $\mathfrak{p}$ good for A, set $L_\mathfrak{p}(A, T) := \det(1 - \varrho_A(\mathrm{Frob}_\mathfrak{p})T)$.
- To these objects, Serre attaches the following data:
  - A compact real Lie subgroup $G \subseteq \mathrm{USp}(2g)$, the *Sato–Tate group*.
  - For each prime $\mathfrak{p}$ good for A, a conjugacy class $y_\mathfrak{p} \in \mathrm{Cl}(G)$ such that

$$\det(1 - y_\mathfrak{p} T) = L_\mathfrak{p}(A, T|\mathfrak{p}|^{-1/2}).$$

- Denote by μ the projection on $\mathrm{Cl}(G)$ of the Haar measure of G

Sato–Tate conjecture

The sequence $\{y_\mathfrak{p}\}_\mathfrak{p}$ is equidistributed with respect to μ.

We will call μ the *Sato–Tate measure*.

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Equidistribution

- Set $Y = \mathrm{Cl}(G)$ and $\mathcal{C}(Y)$ = set of cont. $\mathbb{C}$-valued functions on Y.
- Recall that $\mu\colon \mathcal{C}(Y) \to \mathbb{C}$ is continuous, linear, positive, $\mu(1) = 1$. Write $\int_Y f(y)\mu(y) := \mu(f)$ for $f \in \mathcal{C}(Y)$.
- For $f \in \mathcal{C}(Y)$ and $x > 0$, write

$$\delta_f(x) := \frac{1}{\pi(x)} \sum_{|\mathfrak{p}| \leq x} f(y_\mathfrak{p}).$$

- $\{y_\mathfrak{p}\}_\mathfrak{p}$ is μ-equidistributed on Y

  $\Leftrightarrow$ $\lim_{x\to\infty} \delta_f(x) = \int_Y f(y)\mu(y)$, for every $f \in \mathcal{C}(Y)$.

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  $\lim_{x\to\infty} \delta_\varphi(x) = 0$, for every character $\varphi = \sum_{\chi \neq 1} a_\chi \chi$ of G.

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  the convergence rate of $\{y_{\mathfrak{p}}\}_{|\mathfrak{p}|\leq x}$ towards μ-equidistribution

  by studying:

  the convergence rate of $\delta_{\varphi}(x)$ towards 0

  as $x \to \infty$.

Example

$A/\mathbb{Q}$ elliptic curve without CM $\rightsquigarrow$ $G = \mathrm{USp}(2)$.
Let V denote the standard representation of G. Consider:

$$\varphi_n = \mathrm{Tr}(V^{\otimes n}) - c_n \cdot 1 \quad \text{where} \quad c_n = \langle V^{\otimes n}, 1\rangle = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{1}{n+1}\binom{n}{n/2} & \text{if } n \text{ is even}. \end{cases}$$

Then

$$\delta_{\varphi_n}(x) = \frac{1}{\pi(x)} \sum_{p \leq x} a_p^n - c_n, \qquad \text{where } a_p = \mathrm{Tr}(\rho_A(\mathrm{Frob}_p)).$$

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The connection to L-functions

- For an irreducible character χ of G, consider:

$$L(A,\chi,s) := \prod_{\mathfrak{p} \text{ good}} \det(1 - \varrho(y_{\mathfrak{p}})|\mathfrak{p}|^{-s})^{-1}, \qquad \text{defined for } \Re(s) > 1,$$

  where $\mathrm{Tr}(\varrho) = \chi$.

- $\{y_{\mathfrak{p}}\}_{\mathfrak{p}}$ is μ-equidistributed on Y

  $\Leftrightarrow$ for every nontrivial irreducible character χ of G, the product $L(A,\chi,s)$ extends to a holomorphic nonvanishing function on $\Re(s) \geq 1$.

Assumption

- $L(A,\chi,s)$ is automorphic.

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Two approaches

- First idea: Find an asymptotic upper bound for $|\delta_\varphi(x)|$ as $x \to \infty$.

Theorem (K. Murty)

Let χ be an irreducible nontrivial character of G. Under the Assumption

$$\delta_\chi(x) = O\left(d\frac{\log(N(x+d))\log(x)}{\sqrt{x}}\right).$$

where $N :=$ |conductor of A| and $d := \dim(\chi)[k : \mathbb{Q}]$.

- Second idea: Compute $\int_2^X |\delta_\varphi(x)|^2 dx$ as $X \to \infty$. Not convergent! Define

$$I(\varphi) := \lim_{X\to\infty} \frac{1}{\log(X)} \int_2^X |\delta_\varphi(x)|^2 dx.$$

The *Asymptotic L^2-norm*

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Main result

Theorem(F.–Guitart)

Under the Assumption, for every irreducible nontrivial character χ of G

$$I(\chi) = (2r_\chi + u_\chi)^2 + \sum_{\gamma_\chi \neq 0} \frac{1}{1/4 + \gamma_\chi^2}$$

where:

- $r_\chi := \mathrm{ord}_{s=1/2}(L(A,\chi,s))$,
- $u_\chi := \int_G \chi(g^2)\mu_G(g) = \begin{cases} 0 & \text{if } \exists g \in G \text{ s.t. } \chi(g) \in \mathbb{C} \setminus \mathbb{R}, \\ 1 & \text{if } \rho \text{ is realizable over } \mathbb{R} \\ -1 & \text{otherwise.} \end{cases}$

  is the Frobenius–Schur index of χ,
- γ_χ runs over the non-zero imaginary parts of the zeros of $L(A,\chi,s)$ on the critical region.

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Generalization to non-irreducible characters

- More in general, for $\varphi = \sum_{\chi \neq 1} c_\chi \chi$, we have

$$I(\varphi) = \underbrace{\left(2r_\varphi + u_\varphi\right)^2}_{I_1(\varphi)} + \underbrace{\sum_{\chi \neq 1} \sum_{\gamma_\chi \neq 0} \frac{c_\chi^2}{1/4 + \gamma_\chi^2}}_{I_2(\varphi)} .$$

- Both $I_1(\varphi)$ and $I_2(\varphi)$ can be bounded in terms of N and d.
- For large N, the upper bound for $I_2(\varphi)$ is dominant.
- For small N, the contribution of $I_1(\varphi)$ is still visible.
- The typical example:

$$\varphi = \mathrm{Tr}(V)|_G \rightsquigarrow \begin{cases} \delta_\varphi(x) = \frac{1}{\pi(x)} \sum_{|\mathfrak{p}| \leq x} a_\mathfrak{p} \,, \\ r_\varphi = r_A \quad \text{The analytic rank} \end{cases}$$

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$$\varphi = \mathrm{Tr}(V)|_G \rightsquigarrow \begin{cases} \delta_\varphi(x) = \frac{1}{\pi(x)} \sum_{|\mathfrak{p}| \leq x} a_\mathfrak{p} \,, \\ r_\varphi = r_A \quad \text{The analytic rank} \end{cases}$$

$V =$ the standard representation of $\mathrm{USp}(2g)$.

Generalization to non-irreducible characters

- More in general, for $\varphi = \sum_{\chi \neq 1} c_\chi \chi$, we have

$$I(\varphi) = \underbrace{\left(2r_\varphi + u_\varphi\right)^2}_{I_1(\varphi)} + \underbrace{\sum_{\chi \neq 1} \sum_{\gamma_\chi \neq 0} \frac{c_\chi^2}{1/4 + \gamma_\chi^2}}_{I_2(\varphi)} .$$

- Both $I_1(\varphi)$ and $I_2(\varphi)$ can be bounded in terms of N and d.
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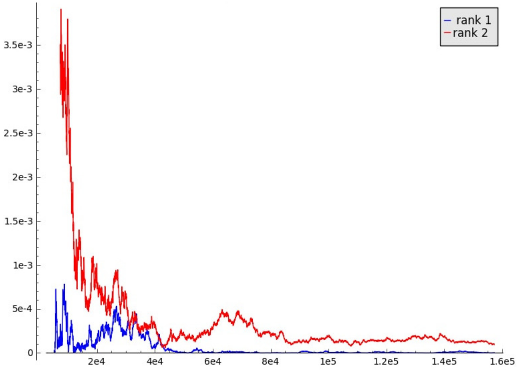
Example I

$$A/\mathbb{Q} \text{ elliptic curve}, \quad N = 390, \quad G = \mathrm{USp}(2), \quad r_A = 1.$$
$$A'/\mathbb{Q} \text{ elliptic curve}, \quad N' = 389, \quad G' = \mathrm{USp}(2), \quad r_{A'} = 2.$$

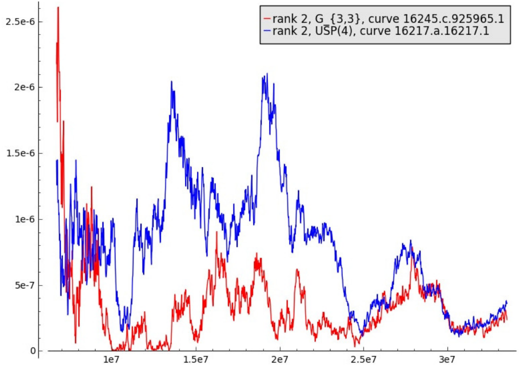
- $I_1(\varphi) = (2 \cdot 1 - 1)^2 = 1, \qquad I_1(\varphi') = (2 \cdot 2 - 1)^2 = 9.$
- Plot of $|\delta_\varphi(x)|^2$ and $|\delta_{\varphi'}(x)|^2$.



Example II

$$A/\mathbb{Q} \text{ ab. surf.}, \quad N = 16217, \quad G = \mathrm{USp}(4), \quad r_A = 2.$$
$$A'/\mathbb{Q} \text{ ab. surf.}, \quad N' = 16245, \quad G' = \mathrm{USp}(2) \times \mathrm{USp}(2), \quad r_{A'} = 2.$$

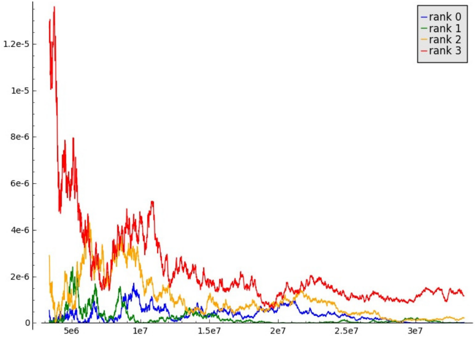
- $I_1(\varphi) = (2 \cdot 2 - 1)^2 = 9, \qquad I_1(\varphi') = (2 \cdot 2 - 2)^2 = 4\,.$
- Plot of $|\delta_\varphi(x)|^2$ and $|\delta_{\varphi'}(x)|^2$:



Example III

$$\begin{array}{llll} A/\mathbb{Q} \text{ ab. surf.}, & N = 62127, & G = \mathrm{USp}(4), & r_A = 0. \\ A'/\mathbb{Q} \text{ ab. surf.}, & N' = 61929, & G' = \mathrm{USp}(4), & r_{A'} = 1. \\ A''/\mathbb{Q} \text{ ab. surf.}, & N'' = 62090, & G'' = \mathrm{USp}(4), & r_A = 2. \\ A'''/\mathbb{Q} \text{ ab. surf.}, & N''' = 62411, & G''' = \mathrm{USp}(4), & r_{A'''} = 3. \end{array}$$

- $I_1(\varphi) = (2 \cdot 0 - 1)^2 = 1, \qquad I_1(\varphi') = (2 \cdot 1 - 1)^2 = 1\,.$
- $I_1(\varphi'') = (2 \cdot 2 - 1)^2 = 9, \qquad I_1(\varphi''') = (2 \cdot 3 - 1)^2 = 25\,.$



Method of proof

- The proof uses the techniques introduced by Sarnak to study:
  The Chebyshev bias in sign of the Frobenius traces of elliptic curves.
  Let $E/\mathbb{Q}$ be an elliptic curve without CM. Sarnak shows:

$$\lim_{X\to\infty} \frac{1}{\log(X)} \int_2^X \frac{\sqrt{x}}{\pi(x)} \sum_{p\leq x} a_p \frac{dx}{x} = -2r_E + 1.$$

- The proof relies on a result by Akbary-Ng-Shahabi:
  If $\delta_\chi(x)$ is 'automorphic', then $\psi(x) = \sqrt{x}|\delta_\chi(x)|$ has a limiting distribution μ_ψ w.r.t dx/x, i.e.

$$\lim_{X\to\infty} \frac{1}{\log(X)} \int_2^X f(\psi(x))\frac{dx}{x} = \int_{\mathbb{R}} f(x)\mu_\psi(x)$$

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