# Compatibility in $p$-adic families

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November 24, 2015

**Abstract**

This is the sixth talk in a series of twelve devoted to the works of G. Kings, D. Loeffler, and S. Zerbes in the Workshop "Arithmetic of Euler systems", celebrated in Benasque in August 2015.

## Contents



# 1 Introduction

The goal of this talk is to construct Beilinson-Flach elements in Hida families which $p$-adically interpolate étale Rankin-Eisenstein classes at level $\Gamma(M,N)$ for a pair of modular forms $f,g$ of weights $k+2, k'+2 \geq 2$ twisted by a cyclotomic variable. These Beilinson-Flach elements project to those constructed by Bertolini, Darmon and Rotger at level $\Gamma_1(N)$ and for weights $k+2 = k'+2 = 2$ (see [BDR15a]). The interpolation property in the case $k+2 = k'+2 = 2$ is shown in [LLZ14], and generalizes the main result of [BDR15b] (in which $f$ is fixed, $g$ varies in a Hida family, and no cyclotomic variable is considered). The proof of the interpolation property in the general case is considered in [KLZ15]. This proof being too long to be reproduced here, we will content ourselves with giving some ideas on the case of a single modular curve (which is treated in [Kin15] by means of a detailed study of the elliptic polylogarithm), that is, we will sketch how Eisenstein-Iwasawa classes interpolate Eisenstein classes.

# 2 Preliminaries on linear algebra

Let $H$ denote the profinite group ${\mathbb{Z}_p}^d$ for $d \geq 1$. We will be interested in the spaces

$$\mathrm{TSym}^k H \quad \text{and} \quad \mathrm{Sym}^k H\,.$$

The first denotes the $\mathbb{Z}_p$-algebra of symmetric $k$-tensors, that is, the space of $\mathfrak{S}_k$-invariants of $H\otimes \overset{k}{\ldots} \otimes H$. In contrast, by the second we denote the $k$th symmetric power of $H$, that is, the space of $\mathfrak{S}_k$-coinvariants of $H\otimes \overset{k}{\ldots} \otimes H$. For $m \leq k$ and $h \in H$, write $h^{[m]} := h^{\otimes m} \in \mathrm{TSym}^m H$. If $(e_1, \ldots, e_d)$ is a basis for $H$, then $(e_1^{[n_1]} \cdots e_d^{[n_d]} | n_1 + \ldots n_d = k)$ is a basis for $\mathrm{TSym}^k H$. We have a $\mathbb{Z}_p$-homomorphism

$$\mathrm{Sym}^k H \to \mathrm{TSym}^k H\,, \qquad e_1^{n_1} \cdots e_d^{n_d} \mapsto k! \cdot e_1^{[n_1]} \cdots e_d^{[n_d]}\,,$$

which becomes an isomorphism after tensoring with $\mathbb{Q}_p$. However, we will keep the distinction between these two spaces, because often we will have to work integrally.

## 2.1 The Clebsch-Gordan map

We wish to define the Clebsch-Gordan map for $k, k' \geq 0$ and $0 \leq j \leq \min\{k, k'\}$

$$\mathrm{CG}^{[\mathrm{k},\mathrm{k}',\mathrm{j}]} \colon \mathrm{TSym}^{k+k'-2j} H \otimes \mathrm{TSym}^j(\wedge^2 H) \to \mathrm{TSym}^k H \otimes \mathrm{TSym}^{k'} H\,.$$

We have an obvious inclusion

$$\mathrm{TSym}^{k+k'-2j} H \subseteq \mathrm{TSym}^{k-j} H \otimes \mathrm{TSym}^{k'-j} H\,.$$

By taking j$th$ powers, the map $\wedge^2 H \to H \otimes H$ that sends $x \wedge y$ to $x \otimes y - y \otimes x$, yields a map

$$\mathrm{TSym}^j(\wedge^2 H) \to \mathrm{TSym}^j H \otimes \mathrm{TSym}^j H\,.$$

The map $\mathrm{CG}^{[\mathrm{k},\mathrm{k}',\mathrm{j}]}$ is obtained as the tensor product of the two previous maps.

## 2.2 The $k$th moment map

Let $(x_1, \ldots, x_d)$ be the dual basis of $(e_1, \ldots, e_d)$, where $x_i : H \to \mathbb{Z}_p$ is seen as a $\mathbb{Z}_p$-valued function on $H$.

Consider the space of $\mathbb{Z}_p$-valued measures on $H$

$$\Lambda(H) := \mathrm{Hom}^{\mathrm{cont}}_{\mathbb{Z}_p}(C(H, \mathbb{Z}_p), \mathbb{Z}_p)\,,$$

where $C(H, \mathbb{Z}_p)$ denotes the space of continuous $\mathbb{Z}_p$-valued functions on $H$.

*Definition* 2.1. The $k$th moment map is the $\mathbb{Z}_p$-algebra homomorphism

$$\mathrm{mom}^k \colon \Lambda(H) \to \mathrm{TSym}^k H\,, \qquad \mathrm{mom}^k(\mu) := \sum_{n_1 + \ldots n_d = k} \mu(x_1^{n_1} \cdots x_d^{n_d}) e_1^{[n_1]} \cdots e_d^{[n_d]}\,.$$

# 3 Étale Eisenstein and Rankin-Eisenstein classes

Let $Y$ denote a modular curve corresponding to a representable moduli problem. It comes equipped with a universal elliptic curve $\pi\colon \mathcal{E} \to Y$. Fix a prime $p$ throughout the talk. We define lisse étale sheaves on $Y[1/p]$:

- $\mathscr{H}_{\mathbb{Z}_p} := R^1\pi_*\mathbb{Z}_p(1) \simeq R^1\pi_*\mathbb{Z}_p{}^\vee$,
- $\mathscr{H}_{\mathbb{Q}_p} := R^1\pi_*\mathbb{Q}_p(1)$,
- $\mathrm{TSym}^k\mathscr{H}_{\mathbb{Z}_p}$,
- $\mathrm{TSym}^k\mathscr{H}_{\mathbb{Q}_p} \simeq \mathrm{Sym}^k\mathscr{H}_{\mathbb{Q}_p}$.

*Remark* 3.1. If $P$ is a geometric point on $Y$ corresponding to an elliptic curve $E$, we should think of the stalk of $\mathscr{H}_{\mathbb{Z}_p}$ at $P$ as the $p$-adic Tate module $M_p(E)$ of $E$. Similarly, we should think of $\mathrm{TSym}^k\mathscr{H}_{\mathbb{Z}_p,P}$ as $\mathrm{TSym}^k M_p(E)$.

For $f = \sum_{n\geq 1} a_n(f)q^n \in S_{k+2}(\Gamma_1(N_f))$ a normalized cuspidal eigenform, $L$ a number field containing $\mathbb{Q}(\{a_n(f)\}_{n\geq 1})$, $N$ divisible by $N_f$, and $\mathfrak{P}$ a prime of $L$ lying over $p$, let:

- $M_{L_\mathfrak{P}}(f)$ be the maximal subspace of $H^1_{et,c}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathrm{Sym}^k\mathscr{H}^\vee_{\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} L_\mathfrak{P}$ on which the Hecke operator $T_\ell$ acts as multiplication by $a_\ell$ for every prime $\ell$.
- $M_{L_\mathfrak{P}}(f)^*$ be the maximal quotient of $H^1_{et,c}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathrm{TSym}^k\mathscr{H}_{\mathbb{Q}_p}(1)) \otimes_{\mathbb{Q}_p} L_\mathfrak{P}$ on which the Hecke operator $T'_\ell$ acts as multiplication by $a_\ell$ for every prime $\ell$.

If $\mathcal{O}_\mathfrak{P}$ denotes the ring of integers of $L_\mathfrak{P}$, then one defines integral versions $M_{\mathcal{O}_\mathfrak{P}}(f)$ and $M_{\mathcal{O}_\mathfrak{P}}(f)^*$ of the previous obtects in the obvious way.

*Definition* 3.2. Let $N \geq 5$, $b \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$, and $k \geq 0$. The *étale Eisenstein class* $\mathrm{Eis}^k_{et,b,N}$ is defined as the image of the motivic Eisenstein class $\mathrm{Eis}^k_{mot,b,N}$ by the étale regulator map

$$H^1_{mot}(Y_1(N), \mathrm{TSym}^k\mathscr{H}_{\mathbb{Q}}(1)) \to H^1_{et}(Y_1(N)_{\mathbb{Z}[1/Np]}, \mathrm{TSym}^k\mathscr{H}_{\mathbb{Q}_p}(1))$$

*Example* 3.3. As we saw in Antonio's talk, for $k = 0$, $H^1_{mot}(Y_1(N), \mathbb{Q}(1)) = \mathcal{O}(Y_1(N))^* \otimes \mathbb{Q}$ and the motivic Eisenstein class $\mathrm{Eis}^0_{mot,b,N}$ is the Siegel unit $g_{0,b/N}$.

Let $f \in S_{k+2}(\Gamma_1(N_f))$ and $g \in S_{k'+2}(\Gamma_1(N_g))$ for $k, k' \geq 0$. To shorten notation, until the end of this §, let us write $Y := Y_1(N)[1/Np]$, where $N$ is an integer divisible by $N_f$ and $N_g$. For $0 \leq j \leq \min\{k, k'\}$, we will be interested in the following maps:

- The Clebsch-Gordan map:

  $$H^1_{et}(Y, \mathrm{TSym}^{k+k'-2j}\mathscr{H}_{\mathbb{Q}_p}(1)) \xrightarrow{\mathrm{CG}^{[k,k',j]}_*} H^1_{et}(Y, \mathrm{TSym}^k\mathscr{H}_{\mathbb{Q}_p} \otimes \mathrm{TSym}^{k'}\mathscr{H}_{\mathbb{Q}_p}(1-j))\,.$$

  At the level of stalks, this is the map defined in 2.1. Indeed, note that in our situation $\wedge^2 H \simeq \det(H) \simeq \mathbb{Q}_p(1)$.

- The push-forward of the diagonal embedding:

$$H^1_{et}(Y, \mathrm{TSym}^k \mathscr{H}_{\mathbb{Q}_p} \otimes \mathrm{TSym}^{k'} \mathscr{H}_{\mathbb{Q}_p}(-j)) \xrightarrow{\Delta_*} H^3_{et}(Y^2, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j)) .$$

  Here, for $\mathcal{A}, \mathcal{B}$ sheaves on $Y$ and $\pi_1, \pi_2 \colon Y^2 \to Y$ the two distinct projections, we write $\mathcal{A} \boxtimes \mathcal{B}$ for the sheaf $\pi_1^* \mathcal{A} \otimes \pi_2^* \mathcal{B}$ on $Y^2$. We then write $\mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p} := \mathrm{TSym}^k \mathscr{H}_{\mathbb{Q}_p} \boxtimes \mathrm{TSym}^{k'} \mathscr{H}_{\mathbb{Q}_p}$.

- There is an edge map coming from the Hochschild-Serre spectral sequence

$$H^3_{et}(Y^2, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j)) \overset{\simeq}{\to} H^1(\mathbb{Z}[\frac{1}{Np}], H^2_{et,c}(Y_1(N)^2_{\overline{\mathbb{Q}}}, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j))) .$$

- Projection to the $(f, g)$-isotypic component

$$H^1(\mathbb{Z}[\frac{1}{Np}], H^2_{et,c}(Y_1(N)^2_{\overline{\mathbb{Q}}}, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j))) \overset{\mathrm{pr}_{f,g}}{\to} H^1(\mathbb{Z}[\frac{1}{Np}], M_{L_{\mathfrak{P}}}(f)^* \otimes M_{L_{\mathfrak{P}}}(g)^*(-j))$$

*Definition* 3.4.

- The *Rankin-Eisenstein class* $\mathrm{Eis}^{f,g,j}_{et,b,N}$ is defined as the image of the étale Eisenstein class $\mathrm{Eis}^{k+k'-2j}_{et,b,N}$ by the concatenation of all the previous maps.

- The *Rankin-Eisenstein class* $\mathrm{Eis}^{[k,k',j]}_{et,b,N}$ at stage

$$H^3_{et}(Y^2, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j)) \simeq H^1(\mathbb{Z}[\frac{1}{Np}], H^2_{et,c}(Y_1(N)_{\overline{\mathbb{Q}}}, \mathrm{TSym}^{[k,k']} \mathscr{H}_{\mathbb{Q}_p}(2-j)))$$

  is defined as the image of the étale Eisenstein class $\mathrm{Eis}^{k+k'-2j}_{et,b,N}$ by the map $\Delta_* \circ \mathrm{CG}^{[k,k',j]}_*$.

# 4 Eisenstein-Iwasawa and Rankin-Iwasawa classes

Recall that as at the beginning of §3, if $Y$ is a modular curve corresponding to a representable moduli problem, we have a universal elliptic curve $\pi \colon \mathcal{E} \to Y$. Let us see $\mathcal{E}$ as a covering of itself by means of

$$[p^r] \colon \mathcal{E}_r := \mathcal{E} \to \mathcal{E}$$

the $p^r$-multiplication map, with $r \geq 1$. Define the pro-system of étale lisse sheaves

$$\mathcal{L} := ([p^r]_* (\mathbb{Z}/p^r\mathbb{Z}))_{r \geq 1} ,$$

which we call the *elliptic polylogarithm.* The transition maps are constructed in the following manner. First consider the composition

$$[p]_* \mathbb{Z}/p^{r+1}\mathbb{Z} \to \mathbb{Z}/p^{r+1}\mathbb{Z} \to \mathbb{Z}/p^r\mathbb{Z} \tag{4.1}$$

of maps of sheaves on $\mathcal{E}_r$, where the first map is the trace map induced by $[p] \colon \mathcal{E}_{r+1} \to \mathcal{E}_r$ and the second is the reduction map. The transition map is now obtained by projecting (4.1) on $\mathcal{E}$ by $[p^r]_*$

$$[p^{r+1}]_* \mathbb{Z}/p^{r+1}\mathbb{Z} \to [p^r]_* \mathbb{Z}/p^r\mathbb{Z} .$$

Write $\mathcal{L}_{\mathbb{Q}_p} := \mathcal{L} \otimes \mathbb{Q}_p$. For a section $t\colon Y \to \mathcal{E}$, define the sheaf of Iwasawa modules

$$\Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle t\rangle) := t^*\mathcal{L}\,,$$
$$\Lambda(\mathscr{H}_{\mathbb{Z}_p}) := \Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle e\rangle)\,,$$

where $e\colon Y \to \mathcal{E}$ denotes the trivial section.

*Remark* 4.1. If $P$ is a geometric point on $Y$ corresponding to an elliptic curve $E$, we should think of the stalk of $\Lambda(\mathscr{H}_{\mathbb{Z}_p})$ at $P$ as the Iwasawa algebra of the $p$-adic Tate module $M_p(E)$ of $E$, that is, the space of $\mathbb{Z}_p$-valued measures on $M_p(E)$. This justifies the notation and terminology used.

*Remark* 4.2. There exist sheafified moment maps

$$\mathrm{mom}^k\colon \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \to \mathrm{TSym}^k \mathscr{H}_{\mathbb{Z}_p}$$

such that if $P$ is a geometric point on $Y$ corresponding to an elliptic curve $E$, then

$$\mathrm{mom}^k_P\colon \Lambda(\mathscr{H}_{\mathbb{Z}_p})_P = \Lambda(M_p(E)) \to \mathrm{TSym}^k \mathscr{H}_{\mathbb{Z}_p,P} = \mathrm{TSym}^k M_p(E)$$

coincides with the $k$th moment map of Definition 2.1.

In Antonio's talk, we have defined the Kato units ${}_c\theta_{\mathcal{E}} \in \mathcal{O}(\mathcal{E}\setminus\mathcal{E}[c])^*$ for $c > 1$ and $(c, 6) = 1$. Observe that

$$H^1_{et}(\mathcal{E}\backslash\mathcal{E}[c], \mathcal{L}(1)) \simeq \varprojlim_r H^1_{et}(\mathcal{E}\backslash\mathcal{E}[c], [p^r]_*(\mathbb{Z}/p^r\mathbb{Z})(1)) \simeq \varprojlim_r H^1_{et}(\mathcal{E}_r\backslash\mathcal{E}_r[cp^r], \mathbb{Z}/p^r\mathbb{Z}(1))\,.$$

Thanks to the norm relations that we saw that Kato units satisfy, if $p \nmid c$ the following limit is well defined

$${}_c\Theta_{\mathcal{E}} := \varprojlim_r \partial_r({}_c\theta_{\mathcal{E}_r}) \in \varprojlim_r H^1_{et}(\mathcal{E}_r \setminus \mathcal{E}_r[cp^r], \mathbb{Z}/p^r\mathbb{Z}(1))\,,$$

where $\partial_r\colon \mathcal{O}(\mathcal{E}_r \setminus \mathcal{E}_r[cp^r])^* \to H^1(\mathcal{E}_r \setminus \mathcal{E}_r[cp^r], \mathbb{Z}/p^r\mathbb{Z}(1))$ is the connecting morphism for the exact sequence

$$1 \to \mu_{p^r} \to \mathbb{G}_m \overset{\cdot p^r}{\to} \mathbb{G}_m \to 1\,.$$

Until the definition of Rankin-Iwasawa class, for $M, N \geq 1$, $M|N$, and $M + N \geq 5$, let $Y$ be the curve $Y(M,N)[1/MNp]$ defined in Kezuka's talk.

*Definition* 4.3. Let $c > 1$ with $(c, 6Np) = 1$ and $b \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$. Let $t_N\colon Y(M,N) \to \mathcal{E} \setminus \mathcal{E}[c]$ be the canonical section of order $N$ (note that it takes values in $\mathcal{E} \setminus \mathcal{E}[c]$ by our choice of $c$). The *Eisenstein-Iwasawa class* ${}_c\mathcal{EI}_t$ is defined as the image of ${}_c\Theta_{\mathcal{E}}$ by the map

$$H^1_{et}(\mathcal{E} \setminus \mathcal{E}[c], \mathcal{L}(1)) \overset{(bt_N)^*}{\to} H^1_{et}(Y, \Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle bt_N\rangle)(1)) \overset{[N]_*}{\to} H^1_{et}(Y, \Lambda(\mathscr{H}_{\mathbb{Z}_p})(1))\,.$$

We will be interested in the following maps:

- The map induced by $\Lambda(\mathscr{H}_{\mathbb{Z}_p}) \to \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \otimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})$

$$H^1(Y, \Lambda(\mathscr{H}_{\mathbb{Z}_p})(1)) \to H^1_{et}(Y, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \otimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(1))\,.$$

- The push-forward of the diagonal embedding $\Delta\colon Y \to Y^2$

$$H^1_{et}(Y, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \otimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(1)) \overset{\Delta_*}{\to} H^3_{et}(Y^2, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(2))\,.$$

- For $a \in \mathbb{Z}/M\mathbb{Z}$, the map

$$H^3_{et}(Y^2, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(2)) \overset{u_{a*}}{\to} H^3_{et}(Y^2, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(2))\,,$$

  where $u_a\colon Y^2 \to Y^2$ is the automorphism that is the identity on the first factor and the map that sends a triple $(E, e_1, e_2)$ to the triple $(E, e_1 + a\frac{N}{M}e_2, e_2)$ on the second factor.

- The edge map

$$H^3_{et}(Y^2, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(2)) \overset{\simeq}{\to} H^1(\mathbb{Z}[\frac{1}{MNp}], H^2_{et,c}(Y(M,N)_{\overline{\mathbb{Q}}}, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(2)))\,.$$

*Definition* 4.4. The *Rankin-Iwasawa class* ${}_c\mathcal{RI}_{M,N,a}$ is defined as the image of the Eisenstein-Iwasawa class ${}_c\mathcal{EI}_{1,N}$ by the concatenation of all the previous maps.

In §5, we will see that Rankin-Iwasawa classes (or even more generally, Beilinson-Flach elements) interpolate Rankin-Eisenstein classes (see Theorem 5.1). To conclude the section, we will see an intermediate result, which shows that Eisenstein-Iwasawa classes (for $M = 1$) interpolate Eisenstein classes.

**Theorem 4.1** (Thm. 4.7.1 of [Kin15]). *For $N \geq 5$, $b \in \mathbb{Z}/N\mathbb{Z} \setminus \{0\}$, and $c > 1$ with $(c, 6Np) = 1$, one has*

$$\mathrm{mom}^k({}_c\mathcal{EI}_{b,N}) = c^2 \operatorname{Eis}^k_{et,b,N} - c^{-k} \operatorname{Eis}^k_{et,cb,N}$$

*as elements of $H^1_{et}(Y_1(N)[1/Np], \mathrm{TSym}^k \mathscr{H}_{\mathbb{Q}_p}(1))$.*

*Sketch of proof.* In the course of the proof, let us write $t\colon Y_1(N) \to \mathcal{E}$ for a section of order $N$, $e\colon Y_1(N) \to \mathcal{E}$ for the trivial section, and $Y := Y_1(N)[1/Np]$. The proof uses the following crucial properties of the elliptic polylogarithm

- For an isogeny $\varphi\colon \mathcal{E} \to \mathcal{E}$, one has $\varphi^*\mathcal{L}_{\mathbb{Q}_p} \simeq \mathcal{L}_{\mathbb{Q}_p}$.
- $e^*\mathcal{L}_{\mathbb{Q}_p} \simeq t^*\mathcal{L}_{\mathbb{Q}_p} \simeq (\Lambda(\mathscr{H}_{\mathbb{Z}_p}\langle t\rangle) \otimes \mathbb{Q}_p) \simeq \prod_{k\geq 0} \mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_p}$.
- There is a multiplication map mult: $\pi^*\mathscr{H}_{\mathbb{Q}_p} \otimes \mathcal{L}_{\mathbb{Q}_p} \to \mathcal{L}_{\mathbb{Q}_p}$.

Consider the following diagram (the first vertical arrow of which we take as a black box[1])

$$
\begin{array}{ccc}
\mathrm{Hom}_Y(\mathcal{H}_{\mathbb{Q}_p}, \prod_{k\geq 1} \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}) & & \\
\downarrow \simeq & & \\
\mathrm{Ext}^1_{\mathcal{E}\setminus\{e\}}(\pi^*\mathcal{H}_{\mathbb{Q}_p}, \mathcal{L}_{\mathbb{Q}_p}(1)) & \xrightarrow{[c]^*} & \mathrm{Ext}^1_{\mathcal{E}\setminus\mathcal{E}[c]}(\pi^*\mathcal{H}_{\mathbb{Q}_p}, [c]^*\mathcal{L}_{\mathbb{Q}_p}(1)) \\
\downarrow t^* & & \downarrow \simeq \\
\mathrm{Ext}^1_Y(\mathcal{H}_{\mathbb{Q}_p}, \prod_{k\geq 0} \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}(1)) & & \mathrm{Ext}^1_{\mathcal{E}\setminus\mathcal{E}[c]}(\pi^*\mathcal{H}_{\mathbb{Q}_p}, \mathcal{L}_{\mathbb{Q}_p}(1)) \\
\downarrow \simeq & & \uparrow \mathrm{mult}_{\mathcal{H}_{\mathbb{Q}_p}} \\
\mathrm{Ext}^1_Y(\mathbb{Q}_p, \mathcal{H}^\vee_{\mathbb{Q}_p} \otimes \prod_{k\geq 0} \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}(1)) & & \mathrm{Ext}^1_{\mathcal{E}\setminus\mathcal{E}[c]}(\pi^*\mathcal{H}_{\mathbb{Q}_p}, \mathcal{L}_{\mathbb{Q}_p} \otimes \pi^*\mathcal{H}_{\mathbb{Q}_p}(1)) \\
\downarrow \simeq & & \uparrow \otimes\pi^*\mathcal{H}_{\mathbb{Q}_p} \\
H^1_{et}(Y, \mathcal{H}^\vee_{\mathbb{Q}_p} \otimes \prod_{k\geq 0} \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}(1)) & & \mathrm{Ext}^1_{\mathcal{E}\setminus\mathcal{E}[c]}(\mathbb{Q}_p, \mathcal{L}_{\mathbb{Q}_p}(1)) \\
\downarrow \mathrm{contr} & & \uparrow \simeq \\
H^1_{et}(Y, \prod_{k\geq 1} \mathrm{Sym}^{k-1} \mathcal{H}_{\mathbb{Q}_p}(1)) & & H^1_{et}(\mathcal{E}\setminus\mathcal{E}[c], \mathcal{L}_{\mathbb{Q}_p}(1)) \\
\downarrow \mathrm{pr}_{k-1} & & \\
H^1_{et}(Y, \mathrm{Sym}^{k-1} \mathcal{H}_{\mathbb{Q}_p}(1)) & &
\end{array}
$$

At the level of stalks the contraction map is defined in the following way

$$\mathrm{contr}\colon H^\vee \otimes \mathrm{Sym}^k H \to \mathrm{Sym}^{k-1} H\,, \qquad h^\vee \otimes h_1 \otimes \cdots \otimes h_k \mapsto \frac{1}{k+1} \sum_{j=1}^{k} h^\vee(h_j) h_1 \otimes \cdots \otimes \hat{h}_j \otimes \cdots \otimes h_k\,.$$

Let pol denote the image of the canonical immersion

$$\mathcal{H}_{\mathbb{Q}_p} \hookrightarrow \prod_{k\geq 0} \mathrm{Sym}^k \mathcal{H}_{\mathbb{Q}_p}$$

by the very first isomorphism in the above diagram and write $t^*\mathrm{pol} := (t^*\mathrm{pol}^k)_{k\geq 1}$. The first step of the proof is to show that if $t = bt_N$, where $t_N$ is the canonical section of order $N$, then the Eisenstein class $\mathrm{Eis}^k_{et,b,N}$ is the image of $t^*\mathrm{pol}^{k+1}$ by the concatenation of the maps in the first column of the the previous diagram.

Note that we had defined Kato elements ${}_c\Theta_{\mathcal{E}} \in H^1_{et}(\mathcal{E}\setminus\mathcal{E}[c], \mathcal{L}_{\mathbb{Q}_p}(1))$. If we detnote by $\mathrm{mult}_{\mathcal{H}_{\mathbb{Q}_p}}$ the concatenation of the maps on the second column, then the second step of the proof consists of establishing the following fundamental relation

$$c^2 \mathrm{pol}|_{\mathcal{E}\setminus\mathcal{E}[c]} - c[c]^*\mathrm{pol} = \mathrm{mult}_{\mathcal{H}_{\mathbb{Q}_p}}({}_c\Theta_{\mathcal{E}})$$

in $\mathrm{Ext}^1_{\mathcal{E}\setminus\mathcal{E}[c]}(\pi^*\mathcal{H}\mathbb{Q}_p, \mathcal{L}_{\mathbb{Q}_p}(1))$. Now the theorem follows from the following two facts:

[1]It follows from the Leray spectral sequence for $\mathcal{L}_{\mathbb{Q}_p}$ and $\pi$, the localization sequence, and the vanishing of $R^i\pi_*\mathcal{L}_{\mathbb{Q}_p}$ except for $i = 2$.

- The concatenation of the maps on the second and first column coincide with the sheafified $k$th moment map $\mathrm{mom}^k$ (once tensored with $\mathbb{Q}_p$); and
- The isomorphism $t^*\mathcal{L}_{\mathbb{Q}_p} \simeq t^*[c]^*\mathcal{L}_{\mathbb{Q}_p}$ is multiplication by $c^k$ on the graded piece $\mathrm{Sym}^k \mathscr{H}_{\mathbb{Q}_p}$.

□

# 5 Beilinson-Flach elements: Projection to $Y_1(N)$

Let $m \geq 1$ and $N \geq 5$. Let $\mu_m^\circ$ be the scheme of primitive $m$th roots of unity, that is, $\mu_m^\circ = \mathrm{Spec}(\mathbb{Z}[\zeta_m])$, where $\zeta_m$ is a primitive $m$th root of unity. In Vivek's talk we have seen that there exists a map[2]

$$\alpha_m \colon Y(m, mN) \to Y_1(N) \times \mu_m^\circ .$$

*Definition* 5.1. We will write ${}_c\mathcal{BF}^{[0]}_{m,N}$ for the image of the Rankin-Iwasawa class ${}_c\mathcal{RI}_{m,mN,1}$ by the map

$$(\alpha_m \times \alpha_m)_* \colon H^3_{et}(Y(m,mN)^2, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(2)) \to H^3_{et}(Y_1(N)^2 \times \mu_m^\circ, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(2))$$

We still need to introduce one more sheaf of Iwasawa modules. Let

$$\mathrm{pr}_r \colon \mathrm{Spec}(\mathbb{Z}[1/p]) \times \mu_{p^r}^\circ \to \mathrm{Spec}(\mathbb{Z}[1/p])$$

the natural projection for $r \geq 1$. Define the pro-étale sheaf

$$\Lambda_\Gamma(-\mathbf{j}) := (\mathrm{pr}_{r*}(\mathbb{Z}/p^r\mathbb{Z}))_{r \geq 1} .$$

The notation is justified by the fact that the stalk of $\Lambda_\Gamma(-\mathbf{j})$ at a geometric point is the Iwasawa algebra $\Lambda_\Gamma$ of the Galois group $\Gamma := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$, equipped with an action of $\Gamma$ by the inverse of the canonical character $\mathbf{j} \colon \Gamma \to \Lambda_\Gamma^*$. There are moment maps

$$\mathrm{mom}^j_\Gamma \colon \Lambda_\Gamma(-\mathbf{j}) \to \mathbb{Z}_p(-j) .$$

The key property of $\Lambda_\Gamma(-\mathbf{j})$ is that it permits to transfer variations on the level to the sheaf side. More precisely, there is an isomorphism

$$\varprojlim_r H^3(Y_1(N)^2 \times \mu_{mp^r}^\circ, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p})(2)) = H^3(Y_1(N)^2 \times \mu_m^\circ, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \otimes \Lambda_\Gamma(2-\mathbf{j})) ,$$

Let us write

$$e' := e'_{\mathrm{ord}} := \lim_{n \to \infty} (U'_p)^{n!}$$

for Ohta's anti-ordinary operator. The operator $(U'_p, U'_p)$ is invertible on the image of $(e', e')$, and the so-called "Second norm relation", seen in Vivek's talk, shows that the inverse limit

$${}_c\mathcal{BF}_{m,N} := \varprojlim_r (U'_p, U'_p)^{-r} (e', e')\, {}_c\mathcal{BF}^{[0]}_{mp^r,N} ,$$

which is an element of

$$(e', e') H^3(Y_1(N)^2 \times \mu_m^\circ, \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \boxtimes \Lambda(\mathscr{H}_{\mathbb{Z}_p}) \otimes \Lambda_\Gamma(2 - \mathbf{j})) ,$$

[2]There, this map was denoted by $t_m$, but in the present talk we reserve this notation for the canonical section of order $m$.

is well defined. The classes ${}_c\mathcal{BF}_{m,N}$ are called *Beilinson-Flach elements*. The following theorem establishes the interpolation property of the Beilinson-Flach elements. It should be seen as a generalization of Theorem 4.1.

**Theorem 5.1** (Thm. 6.3.3 of [KLZ15])**.** *Let $k, k' \geq 0$ and $0 \leq j \leq \min\{k, k'\}$. For a prime $p \geq 3$, $N \geq 1$, $m \geq 1$ and $c > 1$ with $p|N$, $(p, m) = 1$, and $(c, 6mNp) = 1$, we have that*

$$\mathrm{mom}^k \otimes \mathrm{mom}^{k'} \otimes \mathrm{mom}^j_\Gamma({}_c\mathcal{BF}_{m,N}) =$$

$$= (1 - p^j(U'_p, U'_p)^{-1}\sigma_p)(c^2 - c^{-k-k'+2j}\sigma_c^2(\langle c\rangle, \langle c\rangle))\frac{(e', e')\,\mathrm{Eis}^{[k,k',j]}_{et,m,N}}{(-1)^j j!\binom{k}{j}\binom{k'}{j}}\,,$$

*where $\sigma_c$ is the arithmetic Frobenius at $c$ in* $\mathrm{Gal}(\mathbb{Q}(\mu_m)/\mathbb{Q})$.

# 6 Beilinson-Flach elements in Hida families

Set

$$H^1_{\mathrm{ord}}(Np^\infty) := \varprojlim_r e' H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))\,.$$

It is finitely generated and projective over $\Lambda_D := \mathbb{Z}_p[[\mathbb{Z}_p{}^*]]$. For $r \geq 1$, recall the existence of Ohta's twisting map

$$H^1_{\mathrm{ord}}(Np^\infty) \overset{\mathrm{Ohta}}{\to} e' H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t_N\rangle)(1))\,.$$

*Remark* 6.1. For $r = 1$, it is easy to see that Ohta's twisting map is an isomorphism. Indeed, if one defines $\mathcal{E}[p^r]\langle t_N\rangle$ by the cartesian diagram

$$\begin{array}{ccc} \mathcal{E}[p^r]\langle t_N\rangle & \longrightarrow & \mathcal{E}_r := \mathcal{E} \\ \downarrow{\scriptstyle \mathrm{pr}_{r,t}} & & \downarrow{\scriptstyle [p^r]} \\ Y_1(N) & \xrightarrow{t_N} & \mathcal{E} \end{array}\ ,$$

it is not difficult to see that $\mathcal{E}[p^r]\langle t_N\rangle \simeq Y_1(Np^r)$. Set

$$\Lambda(\mathcal{H}_r\langle t_N\rangle) := t_N^*([p^r]_*\mathbb{Z}/p^r\mathbb{Z}) = \mathrm{pr}_{r,t*}(\mathbb{Z}/p^r\mathbb{Z})\,.$$

It follows that

$$H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \Lambda(\mathcal{H}_r\langle t_N\rangle)(1)) \simeq H^1_{et}(\mathcal{E}[p^r]\langle t_N\rangle_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^r\mathbb{Z}(1)) \simeq H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}/p^r\mathbb{Z}(1))\,.$$

By taking limits we get

$$H^1_{et}(Y_1(N)_{\overline{\mathbb{Q}}}, \Lambda(\mathcal{H}_{\mathbb{Z}_p}\langle t_N\rangle)(1)) \simeq H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathbb{Z}_p(1))\,.$$

As we saw in Chris' talk, Ohta's control Theorem states that the compostion of the moment map $\mathrm{mom}^k \circ [N]_!$ with Ohta's twisting map induces an isomorphism

$$H^1_{\mathrm{ord}}(Np^\infty)/I_{k,r} \to e' H^1_{et}(Y_1(Np^r)_{\overline{\mathbb{Q}}}, \mathrm{TSym}^k(\mathcal{H}_{\mathbb{Z}_p})(1))\,, \tag{6.1}$$

where $I_{k,r}$ is the ideal of $\Lambda_D$ generated by $[1 + p^r] - (1 + p^r)^k$. Write $\mathbb{T}_{Np^\infty}$ for the Hecke algebra generated by the Hecke operators $T'_\ell$ acting on $H^1_{\mathrm{ord}}(Np^\infty)$. There are $\Lambda_D$-linear commuting actions of $\mathbb{T}_N p^\infty$ and $G_{\mathbb{Q},S}$, the Galois group of the maximal unramified extension outside the set $S$ of primes dividing $Np$. $\mathbb{T}_N p^\infty$ is a finite projective $\Lambda_D$-algebra.

*Definition* 6.2.

- A *Hida family* $\mathbf{f}$ is any of the finitely many maximal ideals of $\mathbb{T}_N p^\infty$.
- If $\mathbf{f}$ is a Hida family, set
$$M(\mathbf{f})^* := H^1_{\mathrm{ord}}(Np^\infty)_{\mathbf{f}}\,, \qquad \Lambda_{\mathbf{f}} := (\mathbb{T}_N p^\infty)_{\mathbf{f}}\,.$$
- An *arithmetic prime* is a prime ideal $\mathfrak{p}$ of $\Lambda_D$ of height 1 lying over an ideal of the form $I_{k,r}$ for some $k, r$.

Associated to an arithmetic prime $\mathfrak{p}$, there is an eigenform $f_{\mathfrak{p}}$ of level $Np^r$ and weight $k+2$ such that

$$M_{\mathcal{O}_{\mathfrak{P}}}(f_{\mathfrak{p}})^* = M(\mathbf{f})^* \otimes_{\Lambda_{\mathbf{f}}} \mathcal{O}_{\mathfrak{P}}\,,$$

where $\mathfrak{P}$ is a prime of $\mathbb{T}_{Np^\infty}$ above $\mathfrak{p} \subseteq \Lambda_D$ and the tensor product is taken with respect to the projection map

$$\Lambda_{\mathbf{f}} \to \mathcal{O}_{\mathfrak{P}} := \Lambda_{\mathbf{f}}/\mathfrak{P}\,.$$

*Definition* 6.3. For Hida families $\mathbf{f}$ and $\mathbf{g}$ of tame levels $N_f$ and $N_g$, $m \geq 1$ coprime to $p$, and $c > 1$ coprime to $6mN_fN_gp$, we define

$${}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}}_m \in H^1(\mathbb{Z}[\frac{1}{mpN_fN_g}, \mu_m], M(\mathbf{f})^* \otimes M(\mathbf{g})^* \otimes \Lambda_\Gamma(-\mathbf{j}))$$

to be the image of the class ${}_c\mathcal{BF}_{m,N}$ for $N := \mathrm{Lcm}(N_f, N_g)$ under the edge map coming from the Hochschild-Serre spectral sequence, the projection map $Y_1(N)^2 \to Y_1(N_f) \times Y_1(N_g)$, the Künneth formula, and localization at $\mathbf{f}$ and $\mathbf{g}$.

The main and final theorem of this talk is the following.

**Theorem 6.1** (Thm. 8.1.4 of [KLZ15]). *If $f$ and $g$ are ordinary newforms of levels $N_f$ and $N_g$ which are specializations of the Hida families $\mathbf{f}$ and $\mathbf{g}$ of weights $k+2$ and $k'+2$, then for every $0 \leq j \leq \min\{k, k'\}$ the specialization*

$${}_c\mathcal{BF}^{\mathbf{f},\mathbf{g}}_1(f, g, j) \in H^1(\mathbb{Z}[1/pN_fN_g], M_{L_{\mathfrak{P}}}(f)^* \otimes M_{L_{\mathfrak{P}}}(g)^*(-j))$$

*is equal to*

$$\frac{\left(1 - \frac{p^j}{\alpha_f\alpha_g}\right)\left(1 - \frac{\alpha_f\beta_g}{p^{1+j}}\right)\left(1 - \frac{\beta_f\alpha_g}{p^{1+j}}\right)\left(1 - \frac{\beta_f\beta_g}{p^{1+j}}\right)}{(-1)^j j! \binom{k}{j}\binom{k'}{j}} \left(c^2 - \frac{c^{-k-k'+2j}}{\varepsilon_c(f)\varepsilon_c(g)}\right)(\mathrm{Eis}^{f,g,j}_{et,1,N})\,,$$

*where $\alpha_f$, $\beta_f$ are the roots of the Hecke polynomial $X^2 - a_p(f)X + p^{k-1}\varepsilon_p(f)$, and analogously for $\alpha_g$, $\beta_g$.*

*Proof.* Except of three Euler factors, the other factors come from Theorem 5.1 applied to the $p$-stabilizations of $f$ and $g$. Let $N$ be divisible by $N_f$, $N_g$, and $p$. The remaining three Euler factors are obtained by relating the Beilinson-Flach elements ${}_c\mathcal{BF}_{1,N/p}$ and ${}_c\mathcal{BF}_{1,N}$ relative to $f$ and $g$; or equivalently, the Rankin-Iwasawa classes ${}_c\mathcal{RI}_{1,N/p,1}$ and ${}_c\mathcal{RI}_{1,N,1}$ relative to $f$ and $g$. □

**Aknwoledgements.** I would like to thank D. Loeffler and V. Rotger for helpful discussions.

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