Sato–Tate groups of abelian threefolds

Francesc Fité (MIT)

Joint work with Kiran Kedlaya and Andrew Sutherland

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Frobenius traces of elliptic curves

k a number field.

E/k an elliptic curve.

For a prime $\mathfrak{p}$ of good reduction for E, let $q = \mathrm{Nm}(\mathfrak{p})$ and set

$$a_{\mathfrak{p}} := q + 1 - \#E(\mathbb{F}_q)\,.$$

It satisfies

$$Z(E_{\mathfrak{p}}, T) := \exp\left(\sum_{n\geq 1} \#E(\mathbb{F}_{q^n})\frac{T^n}{n}\right) = \frac{1 - a_{\mathfrak{p}}T + qT^2}{(1-T)(1-qT)}\,.$$

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By the Hasse-Weil bound:

$$\overline{a}_{\mathfrak{p}} := \frac{a_{\mathfrak{p}}}{q^{1/2}} \in [-2, 2]\,.$$

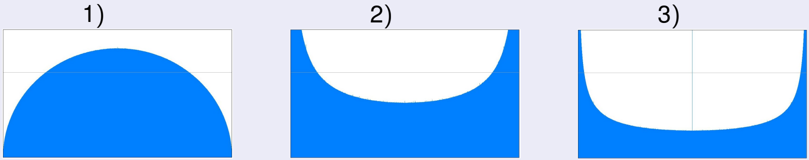
What is the distribution of the $\overline{a}_{\mathfrak{p}}$ on the interval $I = [-2, 2]$?

The Sato–Tate conjecture for elliptic curves

Sato–Tate conjecture for elliptic curves

The sequence $\{\overline{a}_{\mathfrak{p}}\}_{\mathfrak{p}}$ is equidistributed on I w.r.t a measure μ_I given by

1) $\dfrac{1}{2\pi}\sqrt{4-z^2}dz$ if E does not have CM.
2) $\dfrac{1}{\pi}\dfrac{dz}{\sqrt{4-z^2}}$ if E has CM by $M \subseteq k$.
3) $\dfrac{1}{2}\delta_0 + \dfrac{1}{2\pi}\dfrac{dz}{\sqrt{4-z^2}}$ if E has CM by $M \not\subseteq k$.



μ_I from a Haar measure

The measures of the previous slide *come from* real Lie subgroups of

$$\mathsf{SU}(2) := \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in \mathsf{GL}_2(\mathbb{C}) \, : \, a\overline{a} + b\overline{b} = 1 \right\} .$$

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These subgroups are:

1) SU(2) itself.
2) $\mathrm{U}(1) := \left\{ \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix} : u \in \mathbb{C},\ |u| = 1 \right\}$.
3) $N_{\mathrm{SU}(2)}(\mathrm{U}(1)) = \left\langle \mathrm{U}(1), \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$.

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Let G be as in 1), 2), 3). Note that

$$\mathrm{Tr} : G \to I = [-2, 2] \,.$$

Let μ be the Haar measure of G. The measure μ_I satisfies

$$\mathrm{Tr}_*(\mu) = \mu_I \,.$$

Restatement of the conjecture

Define the Sato–Tate group of E as

1) $\mathrm{SU}(2)$ if E does not have CM.
2) $\mathrm{U}(1)$ if E has CM by $M \subseteq k$.
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Recall the numerator of the Zeta function

$$L_{\mathfrak{p}}(E,T) := 1 - a_{\mathfrak{p}} T + qT^2$$

Set

$$\overline{L}_{\mathfrak{p}}(E,T) := L_{\mathfrak{p}}(E, T/q^{1/2}) = 1 - \overline{a}_{\mathfrak{p}} T + T^2 \,.$$

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Sato–Tate conjecture for elliptic curves

The sequence of $\{\overline{L}_{\mathfrak{p}}(E, T)\}_{\mathfrak{p}}$ is equidistributed on the space of charpolys of $\mathrm{ST}(E)$ w.r.t the Haar measure of $\mathrm{ST}(E)$ (projected on this space).

The Sato–Tate group of an abelian variety of dim ≤ 3

Let A/k be an abelian variety of dimension $g \geq 1$.

For a prime ℓ, define

$$T_\ell(A) = \varprojlim_r A[\ell^r](\overline{\mathbb{Q}}), \qquad V_\ell(A) = T_\ell(A) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell .$$

Consider the ℓ-adic representation attached to A

$$\varrho_\ell \colon G_k \to \mathrm{Aut}(V_\ell(A)) .$$

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Denote by $\mathcal{G}_\ell \subseteq \mathrm{GSp}_{2g}/\mathbb{Q}_\ell$ the Zariski closure of the image of ϱ_ℓ.

There is an injection

$$\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{End}_{\mathcal{G}_\ell^0}(V_\ell(A))$$

(by Faltings, in fact an isomorphism).

More conveniently

$$\mathcal{G}_\ell^0 \hookrightarrow \{\gamma \in \mathrm{GSp}_{2g}/\mathbb{Q}_\ell \mid \gamma\alpha\gamma^{-1} = \alpha \text{ for all } \alpha \in \mathrm{End}(A_{\overline{\mathbb{Q}}})\}\,.$$

More accurately

$$\mathcal{G}_\ell \hookrightarrow \bigcup_{\sigma \in G_k} \{\gamma \in \mathrm{GSp}_{2g}/\mathbb{Q}_\ell \mid \gamma\alpha\gamma^{-1} = \sigma(\alpha) \text{ for all } \alpha \in \mathrm{End}(A_{\overline{\mathbb{Q}}})\}\,.$$

Building on the work of many, Banaszak and Kedlaya show that if $g \leq 3$, then the above injection is an equality. From now on assume $g \leq 3$.

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Define the twisted Lefschetz group of A by

$$\mathrm{TL}(A) := \bigcup_{\sigma\in G_k} \{\gamma \in \mathrm{Sp}_{2g}/\mathbb{Q} \,|\, \gamma\alpha\gamma^{-1} = \sigma(\alpha) \text{ for all } \alpha \in \mathrm{End}(A_{\overline{\mathbb{Q}}})\}\,.$$

The Sato–Tate group of A is a maximal compact subgroup of $\mathrm{TL}(A) \times_{\mathbb{Q}} \mathbb{C}$. Denote it $\mathrm{ST}(A)$. It is a subgroup of $\mathrm{USp}(2g)$, well defined *up to conjugacy*.

Sato–Tate conjecture for abelian varieties of dim ≤ 3

Let $\mathfrak{p}$ be a prime of good reduction for A. Define

$$L_{\mathfrak{p}}(A,T) = \det(1-\varrho_\ell(\mathrm{Frob}_{\mathfrak{p}})T \mid V_\ell(A))\,, \qquad \overline{L}_{\mathfrak{p}}(A,T) = L_p(A, T/q^{1/2})\,.$$

Along with $\mathrm{ST}(A)$, one can also define certain $x_{\mathfrak{p}} \in \mathrm{Conj}(\mathrm{ST}(A))$ such that

$$\mathrm{Charpoly}(x_{\mathfrak{p}}) = \overline{L}_{\mathfrak{p}}(A,T)\,.$$

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Sato–Tate conjecture for abelian varieties (Serre)

The sequence $\{x_{\mathfrak{p}}\}_{\mathfrak{p}}$ is equidistributed on $\mathrm{Conj}(\mathrm{ST}(A))$ w.r.t the Haar measure of $\mathrm{ST}(A)$ (projected on this space).

In general the map

$$\mathrm{Conj}(\mathrm{ST}(A)) \to \{\text{Charpolys of } \mathrm{ST}(A)\}\,, \qquad x \mapsto \mathrm{Charpoly}(x)$$

is not injective.

Sato–Tate axioms

$\mathsf{ST}(A)$ satisfies has the following properties:

Hodge condition (ST1). $\mathsf{ST}(A)^0$ contains a Hodge circle and is topologically generated by them (a Hodge circle is the image of a hom $\theta\colon\ \mathsf{U}(1) \to \mathsf{ST}(A)^0$ such that $\theta(u)$ has eigenvalues $u, 1/u$ with multiplicity g).

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Serre condition (ST4). Let F/k be the minimal extension such that $\mathrm{End}(A_F) \simeq \mathrm{End}(A_{\overline{\mathbb{Q}}})$. We call F the endomorphism field of A. Then

$$\mathrm{ST}(A)/\,\mathrm{ST}(A)^0 \simeq \mathrm{Gal}(F/k)\,.$$

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Remarks:

- None of (ST3) and (ST4) are expected for $g \geq 4$.
- Up to conjugacy, 3 subgroups of $\mathrm{USp}(2)$ satisfy the ST axioms.

Sato–Tate groups of abelian surfaces

Theorem (F.-Kedlaya-Rotger-Sutherland; 2012)

- Up to conjugacy in $\mathrm{USp}(4)$, there are 52 Sato–Tate groups of abelian surfaces over number fields.
- The 11 maximal groups (w.r.t finite inclusions) occur as Sato–Tate groups of abelian surfaces over $\mathbb{Q}$.
- The degree of the endomorphism field of an abelian surface (defined over a number field) divides 48.

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Only 34 occur as Sato-Tate groups of abelian surfaces over $\mathbb{Q}$ (FKRS).

There exists a number field k_0 over which all 52 groups arise as the Sato-Tate group of an abelian surface defined over k_0 (F.-Guitart).

All 52 Sato–Tate groups occur as the Sato–Tate group of the Jacobian of a genus 2 curve defined over a number field (FKRS).

Galois endomorphism type

Define the Galois endomorphism type of an abelian variety A/k as the isomorphism class of the $\mathbb{R}$-algebra

$$\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{R} \quad \text{equipped with the action of } \mathrm{Gal}(F/k).$$

Example:

There are three Galois endomorphism types of elliptic curves.

They are $\mathbb{R}$, $\mathbb{C}$ (both equipped with the trivial action), and $\mathbb{C}$ equipped with the action of complex conjugation.

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Theorem (FKRS)

- There are 52 Galois endomorphism types of abelian surfaces over number fields.
- The Sato–Tate group and the Galois endomorphism type of an abelian surface determine each other uniquely.

Comments on the classification $g = 2$

(ST1) allows 6 possibilities for $G^0 \subseteq \mathrm{USp}(4)$ ((ST3) is redundant for $g = 2$).

G^0	$\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{R}$	$N_{\mathrm{USp}(4)}(G^0)/G^0$	$\#\mathcal{A}$
$\mathrm{USp}(4)$	$\mathbb{R}$	C_1	1
$\mathrm{SU}(2) \times \mathrm{SU}(2)$	$\mathbb{R} \times \mathbb{R}$	C_2	2
$\mathrm{SU}(2) \times \mathrm{U}(1)$	$\mathbb{R} \times \mathbb{C}$	C_2	2
$\mathrm{U}(1) \times \mathrm{U}(1)$	$\mathbb{C} \times \mathbb{C}$	D_4	8
$\mathrm{SU}(2)_2$	$\mathrm{M}_2(\mathbb{R})$	$\mathrm{O}(2)$	10
$\mathrm{U}(1)_2$	$\mathrm{M}_2(\mathbb{C})$	$\mathrm{SO}(3) \times C_2$	32
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- $\mathcal{A} =$ set of finite subgroups of $N_{\mathrm{USp}(4)}(G^0)/G^0$ for which (ST2) is satisfied.

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- 3 of the groups in the case $G^0 = \mathrm{U}(1) \times \mathrm{U}(1)$ do not satisfy (ST4):
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  - $G/G^0 \simeq \mathrm{Gal}(F/k)$

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Sato–Tate groups for $g = 3$

Theorem(F.-Kedlaya-Sutherland; 2019)

- Up to conjugacy in $\mathrm{USp}(6)$, there are 410 Sato–Tate groups of abelian threefolds over number fields.
- The 33 maximal groups (w.r.t finite inclusions) occur as Sato–Tate groups of abelian threefolds over $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{3})$.
- The degree of the endomorphism field of an abelian threefold (defined over a number field) divides 192, 336, or 432.

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That 192 and 336 can be achieved was shown in F.-Lorenzo-Sutherland.

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How many over $\mathbb{Q}$?

Is there a k_0 over which all 410 groups can be realized?

Do they all occur among Jacobians of genus 3 curves?

De they all occur among principally polarized abelian threefolds?

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$\mathrm{U}(1) \times \mathrm{U}(1)_2$	$\mathbb{C} \times \mathrm{M}_2(\mathbb{C})$	$C_2 \times \mathrm{SO}(3) \times C_2$	122
$\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$	$\mathbb{R} \times \mathbb{R} \times \mathbb{R}$	S_3	4
$\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$	$\mathbb{C} \times \mathbb{C} \times \mathbb{C}$	$(C_2 \times C_2 \times C_2) \rtimes S_3$	33
$\mathrm{SU}(2)_3$	$\mathrm{M}_3(\mathbb{R})$	$\mathrm{SO}(3)$	11
$\mathrm{U}(1)_3$	$\mathrm{M}_3(\mathbb{C})$	$\mathrm{PSU}(3) \rtimes C_2$	171

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