Endomorphism algebras of geometrically split abelian surfaces over $\mathbb{Q}$

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A conjecture

- Let F be a number field and A/F an abelian variety.
- Call $\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}$ the *algebra of* $\overline{\mathbb{Q}}$*-endomorphisms of* A.
- For natural numbers g and d, consider

$$\mathcal{A}_{g,d} = \{\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \,|\, A/F \text{ has dimension } g \text{ and } [F : \mathbb{Q}] = d\}/\simeq$$

Conjecture (Coleman)

The set $\mathcal{A}_{g,d}$ is finite.

Example

For $g = 1$ and $d = 1$: $\mathcal{A}_{1,1}$ is $\mathbb{Q}$ and the 9 quadratic imaginary fields of class number 1.

An open question

- We are interested in the case $g = 2$ and $d = 1$.
- Set
$$\mathcal{A}_{2,1} = \mathcal{A}_{2,1}^{\mathrm{simple}} \cup \mathcal{A}_{2,1}^{\mathrm{split}},$$
where
$$\mathcal{A}_{2,1}^{\mathrm{simple}} = \{\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \in \mathcal{A}_{2,1} \,|\, A_{\overline{\mathbb{Q}}} \text{ simple } \}$$
$$\mathcal{A}_{2,1}^{\mathrm{split}} = \{\mathrm{End}(A_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q} \in \mathcal{A}_{2,1} \,|\, A_{\overline{\mathbb{Q}}} \sim E \times E'\}$$
- Whether $\#\mathcal{A}_{2,1}^{\mathrm{simple}} < \infty$ is widely open.
- The finiteness of $\mathcal{A}_{2,1}^{\mathrm{split}}$ is known. But what is $\mathcal{A}_{2,1}^{\mathrm{split}}$?

Main result

Theorem (F.-Guitart)

The set $\mathcal{A}_{2,1}^{\mathrm{split}}$ is made of:

i) $\mathbb{Q}\times\mathbb{Q}$, $\mathbb{Q}\times M$, or $M_1\times M_2$, where M, $M_1\not\simeq M_2$ are quadratic imaginary fields of class number 1;

ii) $\mathrm{M}_2(\mathbb{Q})$ or $\mathrm{M}_2(M)$, where M is a quadratic imaginary field with class group 1, $\mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ distinct from

$$\mathbb{Q}(\sqrt{-195}),\mathbb{Q}(\sqrt{-312}),\mathbb{Q}(\sqrt{-340}),\mathbb{Q}(\sqrt{-555}),\mathbb{Q}(\sqrt{-715}),\mathbb{Q}(\sqrt{-760}).$$

- Case i): $1+9+\binom{9}{2}=46$ possibilities.
- Case ii): $1+9+18+(24-6)=46$ possibilities.