GROUPS DEFINABLE IN ORDERED VECTOR SPACES OVER ORDERED DIVISION RINGS

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Abstract. Let $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda\in D}\rangle$ be an ordered vector space over an ordered division ring D, and $G = \langle G, \oplus, e_G\rangle$ an n-dimensional group definable in $\mathcal{M}$. We show that if G is definably compact and definably connected with respect to the t-topology, then it is definably isomorphic to a 'definable quotient group' U/L, for some convex $\bigvee$-definable subgroup U of $\langle M^n, +\rangle$ and a lattice L of rank n. As two consequences, we derive Pillay's conjecture for a saturated $\mathcal{M}$ as above and we show that the o-minimal fundamental group of G is isomorphic to L.

1. Introduction

By [Pi1], we know that every group definable in an o-minimal structure can be equipped with a unique definable manifold topology that makes it into a topological group, called t-topology. **We fix a sufficiently saturated ordered vector space $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda\in D}\rangle$ over an ordered division ring** $D = \langle D, +, \cdot, <, 0, 1\rangle$. Definability is always meant in $\mathcal{M}$ with parameters. By [vdD, Chapter 1, (7.6)], $\mathcal{M}$ is o-minimal. In this paper we study definable groups and prove an 'o-minimal analogue' of the following classical result from the theory of topological groups (see [Pon, Theorem 42], for example):

Fact 1.1. *Any compact connected commutative locally Euclidean group is (as a topological group) isomorphic to a direct product of copies of* $\langle \mathbb{R}, +\rangle/\mathbb{Z}$.

A reasonable model theoretic analogue of this fact should have its assumptions weakened (to their definable versions), since in the non-archimedean extension $\mathcal{M}$ of $\langle \mathbb{R}, +\rangle$ compactness and connectedness almost always fail. Also, caution is needed in order to state a *definable* version of the conclusion, since: i) $\mathbb{Z}$ is not definable in any o-minimal structure and therefore $M/\mathbb{Z}$ is not a priori a definable object, ii) no $[0, a)$, $a \in M$, can serve as a fundamental domain for $M/\mathbb{Z}$, as it cannot contain a representative for the $\mathbb{Z}$-class of infinitely large elements, and iii) we cannot always expect G to be a direct product of 1-dimensional definable subgroups of it, known by examples in [Str] (see also [PeS]).

Let us start with some definitions. M is equipped with the order topology. $M^n = \langle M^n, +\rangle$ is the topological group whose group operation is defined pointwise, that has $\bar{0} = (0, \ldots, 0)$ as its unit element, and whose topology is the product topology. If L is a subgroup of M^n, we denote by E_L the equivalence relation on

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M^n induced by L, namely, $xE_Ly \Leftrightarrow x-y \in L$. For $U \subseteq M^n$, we let $E^U_L := E_L \restriction_{U\times U}$ and $U/L := U/E^U_L$. The elements of U/L are denoted by $[x]^U_L$, $x \in U$. If $U \leqslant M^n$ is a subgroup of M^n, then it is a topological group equipped with the subspace topology. If, moreover, $L \leqslant U$, then $U/L = \langle U/L, +_{U/L}, [\bar{0}]^U_L \rangle$ is the *quotient topological group*, whose topological and group structure are both induced by the canonical surjection $\pi : U \to U/L$. If $S \subseteq U$ is a complete set of representatives for E^U_L (that is, it contains exactly one representative for each equivalence class), then the bijection $U/L \ni [x]^U_L \mapsto x \in S$ induces on S a topological group structure $\langle S, +_S \rangle$:

(i) $x +_S y = z \ \Leftrightarrow\ [x]^U_L +_{U/L} [y]^U_L = [z]^U_L \ \Leftrightarrow\ x + y\, E^U_L\, z$, and

(ii) $A \subseteq S$ is open in the *quotient topology on S* if and only if $\pi^{-1}(A)$ is open in U.

Definition 1.2. Let $U \subseteq M^n$ and $L \leqslant M^n$. Then U/L is said to be *a definable quotient* if there is a definable complete set $S \subseteq U$ of representatives for E^U_L. If, in addition, $L \leqslant U \leqslant M^n$ and for some S as above $+_S$ is definable, then the topological group U/L is called *a definable quotient group*.

Convention. We identify a definable quotient group U/L with $\langle S, +_S \rangle$, for some fixed, definable complete set of representatives S for E^U_L, via the bijection $U/L \ni [x]^U_L \mapsto x \in S$.

That is, a definable quotient group U/L is a definable group and, thus, it can be equipped with the t-topology. As it is shown in Claim 2.7, the t-topology on U/L coincides with the quotient topology on it in the case where L is a 'lattice'. Let us define the notion of a lattice. The abelian subgroup of M^n generated by the elements $v_1, \ldots, v_m \in M^n$ is denoted by $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$. If $v_1, \ldots, v_m$ are $\mathbb{Z}$-linearly independent, then the free abelian subgroup $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$ of M^n is called *a lattice of rank m*.

Moreover, it is shown in Claim 2.7 that if L is a lattice and U/L is a definable quotient, then U can be generated by some definable subset H of it, that is, it has form $U = \bigcup_{k<\omega} H^k$, where $H^k := \underbrace{H + \ldots + H}_{k-\text{times}}$. Such a group U is called '$\bigvee$-definable' in [PeSt], 'locally definable' in [Ed2], and 'Ind-definable' in [HPP].

Definition 1.3 ([PeSt])**.** Let $\{X_k : k < \omega\}$ be a collection of definable subsets of M^n. Assume that $U = \bigcup_{k<\omega} X_k$ is equipped with a binary map $\cdot$ so that $\langle U, \cdot \rangle$ is a group. U is called *a $\bigvee$-definable group* if, for all $i, j < \omega$, there is $k < \omega$, such that $X_i \cup X_j \subseteq X_k$ and the restriction of $\cdot$ to $X_i \times X_j$ is a definable function into M^n.

The reader is referred to [PeSt] for a more detailed discussion of $\bigvee$-definable groups. The main fact about a $\bigvee$-definable group $U = \bigcup_{k<\omega} X_k$ that we use here is that every definable subset of U is contained in some X_k, $k < \omega$, by use of compactness.

Our main result is the following.

Theorem 1.4. *Let $G = \langle G, \oplus, e_G \rangle$ be an n-dimensional group definable in $\mathcal{M}$, which is definably compact and definably connected with respect to the t-topology. Then G is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup $U \leqslant M^n$ and a lattice $L \leqslant U$ of rank n.*

For a definition of convexity see Definition 3.1(i).

Let us point out that in the case where $\mathcal{M}$ is archimedean, Theorem 1.4 has independently been proved in [Ons].

Theorem 1.4 has two corollaries.

Proposition 5.1 (Pillay's conjecture)**.** *Let G be an n-dimensional group definable in $\mathcal{M}$, definably compact and definably connected with respect to the t-topology. Then, there is a smallest type-definable subgroup G^{00} of G of bounded index such that G/G^{00}, when equipped with the logic topology, is a compact Lie group of dimension n.*

Proposition 6.13. *Let G be as in Theorem 1.4. Then the o-minimal fundamental group of G is isomorphic to L.*

Pillay's conjecture was raised in [Pi2] for definably compact groups definable in *any* o-minimal structure. For o-minimal expansions of (real closed) fields, a positive answer was obtained in [HPP]. Earlier, it was shown in [EdOt] that for a group G satisfying the assumptions of the conjecture and definable over an o-minimal expansion of a field, the following hold: (i) the o-minimal fundamental group of G is equal to $\mathbb{Z}^n$, and (ii) the k-torsion subgroup of G is equal to $(\mathbb{Z}/k\mathbb{Z})^n$. Theorem 1.4 and Proposition 6.13 show that (i) and (ii) are true in the present context as well.

Structure of the paper. In Section 2 we fix terminology and recall basic properties of groups definable in $\mathcal{M}$. We also include a discussion on definable quotients and $\bigvee$-definable groups.

In Section 3, we study definability in $\mathcal{M}$ and prove several lemmas to be used in the proof of Theorem 1.4. Among others, we show that [PePi, Corollary 3.9] is true in our context as well, namely, the union of any two non-generic definable subsets of G is non-generic.

Section 4 contains the proof of Theorem 1.4. En route, we show that *any* m-dimensional group definable in $\mathcal{M}$ is locally isomorphic to M^m.

In Section 5, we apply our analysis to define G^{00} and prove Proposition 5.1.

In Section 6, we prove Proposition 6.13.

On notation. English letters denote elements or tuples from M. The Greek letters λ, μ, ν, ξ denote elements or matrices over D. The rest of Greek letters are used to denote paths. The letter ε is also used to denote 'small' elements or tuples from M.

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2. Preliminaries

We assume throughout some familiarity with the basics of o-minimality. (For a standard reference see [vdD].) Definability is always meant in $\mathcal{M}$ with parameters.

For the beginning of this section $\mathcal{M}$ could be any saturated o-minimal structure. A group $G = \langle G, \oplus, e_G \rangle$ is said to be definable if both its domain G and the graph of its group operation are definable subsets of M^n and M^{3n}, for some n, respectively. A topological group is a group equipped with a topology in a way that makes its addition and inverse operation continuous. An isomorphism between two topological groups G and G' is at the same time a group isomorphism and a topological homeomorphism between G and G'.

For the rest of this section, let $G = \langle G, \oplus, e_G \rangle$ be a definable group with $G \subseteq M^n$ and $\dim(G) = m \leq n$.

A definable manifold topology on G is a topology on G satisfying the following: there is a finite set $\mathcal{A} = \{\langle S_i, \phi_i \rangle : i \in J\}$ such that

(i) for each $i \in J$, S_i is a definable open subset of G and $\phi : S_i \to M^m$ is a definable homeomorphism between S_i and $K_i := \phi(S_i) \subseteq M^m$,

(ii) $G = \cup_{i \in J} S_i$, and

(iii) for all $i, j \in J$, if $S_i \cap S_j \neq \emptyset$, then $S_{ij} := \phi_i(S_i \cap S_j)$ is a definable open subset of G and $\phi_j \circ \phi_i^{-1} \restriction_{S_{ij}}$ is a definable homeomorphism onto its image.

We fix our notation for a definable manifold topology on G as above. Moreover, we refer to each ϕ_i as *a chart map*, to each $\langle S_i, \phi_i \rangle$ as *a definable chart on* G, and to $\mathcal{A}$ as *a definable atlas on* G. If all of S_i and ϕ_i, $i \in J$, are A-definable, for some $A \subseteq M$, we say that G admits an A-definable manifold structure.

The main result in [Pi1] is the following.

Fact 2.1. *There is a unique definable manifold topology that makes G into a topological group. We refer to this topology as the* t-topology (on G).

Remark 2.2. (i) Whenever $f : K \to K'$ is a definable bijection between two definable subsets of cartesian powers of M, and $K = \langle K, \star, e \rangle$ is a definable group, f induces on K' a definable group structure $\langle K', \circ, f(e) \rangle$, where $\circ$ is defined as follows: $x \circ y = f\big(f^{-1}(x) \star f^{-1}(y)\big)$. Clearly, f is a definable group isomorphism between K and K'. Moreover, if K is a topological group, f induces on K' a group topology that makes f a definable isomorphism between topological groups.

(ii) By uniqueness of the t-topology, a definable group isomorphism between two definable groups also preserves their associated t-topologies, and thus it is a definable isomorphism between the corresponding topological groups.

We omit bars from tuples in M^n. Let $X \subseteq M^n$ be an A-definable set, for some set of parameters $A \subseteq M$. Then $a \in X$ is called a *dim-generic element of X over A* if $\dim(a/A) = \dim(X)$. If $A = \emptyset$, a is called a *dim-generic element of X*. A definable set $V \subseteq X$ is called *large in* X if $\dim(X \setminus V) < \dim(X)$. Equivalently, V contains all dim-generic elements of X over A, for any A over which X and V are defined. We freely use any properties of dim-generic elements of definable groups from [Pi1].

For the rest of this section $\mathcal{M}$ could be any saturated o-minimal expansion of an ordered group. We make a few comments about the existence of the two topologies on G, the t-topology on the one hand, and the subspace topology induced by M^n, henceforth called $\mathcal{M}$*-topology*, on the other. First, $\oplus$ is continuous with respect to the t-topology, and $+ \restriction_A$ with respect to the $\mathcal{M}$-topology, for $A = \{(x, y) \in G \times G : x + y \in G\}$. Moreover, by [Pi1], the two topologies coincide on a large open subset W^G of G. For $a \in M^n$ and $r > 0$ in M, we denote by $\mathcal{B}^n_a(r)$ the open n-box centered at a of size r,

$$\mathcal{B}^n_a(r) := a + (-r, r)^n = \{a + \varepsilon : \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in M^n,\ \varepsilon_i \in (-r, r)\},$$

whereas for $a \in G$, by a *t-neighborhood* V_a of a (in G) we mean a definable open neighborhood of a in G with respect to the t-topology. We omit the index 'n' from $\mathcal{B}^n_a(r)$ when it is clear that $a \in M^n$. Note that if $\dim(G) = n$ and $a \in W^G$, then for sufficiently small r, $\mathcal{B}_a(r)$ is also a t-neighborhood of a in G.

In general, we distinguish between topological notions when taken with respect to the product topology of M^n and when taken with respect to the t-topology on G, by adding an index 't' in the latter case. For example, we write $\overline{A}^t$, $\mathrm{Int}(A)^t$, $\mathrm{bd}(A)^t = \overline{A}^t \setminus \mathrm{Int}(A)^t$ to denote, respectively, the closure, interior and boundary of a set $A \subseteq G$ with respect to the t-topology. Similarly, $A \subseteq G$ is called 't-open', 't-closed', or 't-connected', if it is definable and, respectively, open, closed, or definably connected with respect to the t-topology. We call a function $f : M^n \to G$ *t-continuous* if it is continuous with respect to the t-topology in the range. Accordingly, $\lim^t_{x \to x_0} f(x)$ denotes the limit of f with respect to the t-topology in the range. Definable compactness of a definable group G is always meant with respect to the t-topology, that is $\big($[PeS]$\big)$: for every definable t-continuous embedding $\sigma : (a, b) \subseteq M \to G$, $-\infty \leq a < b \leq \infty$, there are $c, d \in G$ such that $\lim^t_{x \to a^+} \sigma(x) = c$ and $\lim^t_{x \to b^-} \sigma(x) = d$. By a *$t$-path* we mean a definable t-continuous function $\gamma : [p, q] \to G$, $p, q \in M$, $p \leq q$, and by a *path (in M^n)*, just a definable continuous function $\gamma : [p, q] \to M^n$, $p, q \in M$, $p \leq q$. A *(t-)loop* is then a (t-)path γ with $\gamma(p) = \gamma(q)$. A concatenation of two (t-)paths $\gamma : [p, q] \to M^n$ (G) and $\delta : [r, s] \to M^n$ (G) with $\gamma(q) = \delta(r)$ is a (t-)path $\gamma \vee \delta : [p, q + s - r] \to M^n$ (G) with:

$$(\gamma \vee \delta)(t) = \begin{cases} \gamma(t) & \text{if } t \in [p, q], \\ \delta(t - q + r) & \text{if } t \in [q, q + s - r]. \end{cases}$$

We often let the domain of a (t-)path have the form $[0, p]$, for $0 \leq p \in M$. Since a (t-)path $\gamma : [p, q] \to M^n$ can be reparametrized as $\delta : [0, q - p] \to M^n$, where $\forall t \in [0, q - p]$, $\delta(t) = \gamma(t + p)$, this convention is at no loss of generality. The image of a (t-)path γ is denoted by $\mathrm{Im}(\gamma)$. Finally, a definable subset of M^n (G) is called *(t-)path-connected* if any two points of it can be connected by a (t-)path.

Notice the systematic omittance of the words 'definable' or 'definably' in our terminology.

Remark 2.3. *If G is t-connected, then it is t-path-connected.* In o-minimal expansions of ordered groups, definable connectedness is equivalent to definable path-connectedness. Recall, G can be covered by finitely many t-open sets S_i, that can be taken to be t-connected, each of which is homeomorphic to a definably connected and, thus, path-connected subset of M^m. The homeomorphisms imply that the S_i's are t-path-connected, and thus so is G.

Of course, G, as a definable subset of M^n, has finitely many path-connected components.

The following sets are going to be important in our proof of Theorem 1.4.

Definition 2.4. Let W^G be a fixed definable large t-open subset of G on which the $\mathcal{M}$- and t- topologies coincide. Let

$$V^G := \{a \in G : \text{there is a } t\text{-neighborhood}\, V_a \text{ of } a \text{ in}\, G,$$

$$\text{such that}\, \forall x, y \in V_a,\ x \ominus a \oplus y = x - a + y\} \cap W^G.$$

Lemma 2.5. *(i) V^G is definable.*

(ii) V^G is t-open, and thus also open in the $\mathcal{M}$-topology of G.

Proof. (i) Recall that G admits a definable atlas $\mathcal{A} = \{\langle S_i, \phi_i\rangle : i \in J\}$. Thus, for every element $a \in S_i \subseteq G$, the existence of a t-neighborhood V_a of a in G amounts to the existence of some $r \in M$ such that the image of a under $\phi_i : S_i \to K_i$ belongs to the open m-box $\mathcal{B}_{\phi_i(a)}(r) \subseteq K_i$ in M^m.

(ii) Let $v \in V^G$ and a t-neighborhood $V_v \subseteq G$ contain v such that $\forall x, y \in V_v$, $x \ominus v \oplus y = x - v + y$. By the definable manifold structure of G and Remark 2.2, we can assume that $V_v = \mathcal{B}^m_v(r)$ for some $r > 0$ in M. We claim that $\forall u \in \mathcal{B}^m_v(r)$, $u \in V^G$. To see that, let $u \in \mathcal{B}^m_v(r)$ and pick $\delta > 0$ in M such that $\mathcal{B}^m_u(\delta) \subseteq \mathcal{B}^m_v(r)$. Let $x, y \in \mathcal{B}^m_u(\delta)$. Then, $v + x - u \in \mathcal{B}^m_v(r)$ and

$$(v + x - u) \ominus v \oplus u = v + x - u - v + u = x.$$

Therefore, $x \ominus u = (v + x - u) \ominus v$. It follows that

$$x \ominus u \oplus y = (v + x - u) \ominus v \oplus y = v + x - u - v + y = x - u + y.$$

□

2.1. **Definable quotients and $\bigvee$-definable groups.** First, a general statement about quotient topological groups:

Lemma 2.6. *Let $L \leqslant U \leqslant M^n$, and $S \subseteq U$ a complete set of representatives for E^U_L. Let $R \subseteq S$ be open in U. Then, for any $D \subseteq R$, D is open in U if and only if D is open in the quotient topology on S.*

Proof. First, we claim that every $A \subseteq S$ open in U is open in the quotient topology on S. Let $A \subseteq S$ be open in U. We need to show that $\pi^{-1}(A)$ is open in U. But $\pi^{-1}(A) = \bigcup_{x\in L}(x + A)$. Since $\langle U, +, 0\rangle$ is a topological group, we have that for all $x \in L$, $x + A$ is open in U. Thus, $\bigcup_{x\in L}(x + A)$ is open in U.

Now let $R \subseteq S$ be open in U, and $D \subseteq R$. The left-to-right direction is given by the previous paragraph. For the right-to-left one, assume D is open in the quotient topology on S, that is, $\pi^{-1}(D) = \bigcup_{x\in L}(x + D)$ is open in U. Since R is also open in U, it suffices to show

$$D = \pi^{-1}(D) \cap R.$$

$D \subseteq \pi^{-1}(D) \cap R$ is clear. Now, let $a \in \pi^{-1}(D) \cap R$. We have $a = x + d = r$, for some $x \in L$, $d \in D$ and $r \in R$. Thus, $d - r \in L$. Since S is a complete set of representatives for E^U_L, and $d, r \in S$, we have $d = r$. Thus, $x = 0$ and $a = d \in D$. □

Claim 2.7. *Let $L \leqslant U \leqslant M^n$, with L a lattice of rank $m \leq n$. Suppose U/L is a definable quotient, and let $S \subseteq U$ be a definable complete set of representatives for E^U_L. Then:*

(i) U is a $\bigvee$-definable group,

(ii) U/L is a definable quotient group, and

(iii) the quotient topology on S coincides with the t-topology on S.

Proof. (i) We have, $\forall x \in U, \exists y \in S, x - y \in L$. Let $L = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$, and for each $k < \omega$,

$$L_k := \{l_1v_1 + \ldots + l_nv_n \in L : -k \leq l_i \leq k\}$$

and

$$U_k := \{x \in M^n : \exists y \in S,\, x - y \in L_k\} = S + L_k.$$

Clearly, all L_k and U_k are definable. Moreover, $U = \bigcup_{k<\omega} U_k$. Since $\forall k$, $U_k \subseteq U_{k+1}$, it is easy to see that U is $\bigvee$-definable.

(ii) Since $U = \bigcup_{k<\omega} U_k$ is $\bigvee$-definable and $S + S \subseteq U$, there must be some $K < \omega$ such that $S + S \subseteq U_K$. It follows that $+_S$ is definable, since $\forall x, y, z \in S$, $x +_S y = z \Leftrightarrow x + y\, E_L^{U_K}\, z \Leftrightarrow x + y - z \in L_K$.

(iii) Since $\langle S, +_S\rangle$ is a topological group with respect to the quotient topology as well as with respect to the t-topology, it suffices to show that the two topologies coincide on a large subset Y of S. Let W^S be as in Definition 2.4, that is, W^S is a large open subset of S where the t-topology coincides with the subspace topology induced by M^n (or by U).

Subclaim. *There is a definable set $R \subseteq S$ which is open in U and large in S.*

Proof of Subclaim. For a topological space A and a set $B \subseteq A$, let us denote by $\mathrm{Int}_A(B)$ the interior of B in A. For $k < \omega$, let $X_k := \mathrm{Int}_U(U_k)$. Since the topology on U is the subspace topology by M^n, each X_k is definable.

We first show that S is contained in some X_k. By compactness, it suffices to show that $U = \bigcup_{k<\omega} X_k$. To see that, first note $U = \mathrm{Int}_U(U)$. That is, for any $x \in U$, there is a definable open set $X \subseteq U$ containing x. But, for some $k < \omega$, $X \subseteq U_k$. Thus, $x \in \mathrm{Int}_U(U_k) = X_k$.

Now, let $k < \omega$ so that $S \subseteq X_k$. Since X_k is open in U, we have $\mathrm{Int}_{X_k}(S) = \mathrm{Int}_U(S)$. By [vdD, Chapter 4, Corollary (1.9)], $\dim\big(S \setminus \mathrm{Int}_{X_k}(S)\big) < \dim(S)$, that is, $\mathrm{Int}_{X_k}(S)$ is large in S. Let $R := \mathrm{Int}_U(S) = \mathrm{Int}_{X_k}(S)$. Then R is definable, large in S and open in U. □

Let R be as in Subclaim. Then $Y := R \cap W^S \subseteq M^n$ is a large subset of S. Let $D \subseteq Y$. We have: $D \subseteq R$ is open in the quotient topology on S if and only if (by Lemma 2.6) D is open in U if and only if $D \subseteq W^S$ is open in the t-topology on S. □

3. Definability in $\mathcal{M}$

We discuss here some facts about the o-minimal theory $Th(\mathcal{M})$ of ordered vector spaces over ordered division rings, and set up the scene for the proof of Theorem 1.4 in the next section. Following [vdD, Chapter 1, §7], a *linear (affine) function* on $A \subseteq M^n$ is a function $f : A \to M$ of the form $f(x_1, \dots, x_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n + a$, for some fixed $\lambda_i \in D$ and $a \in M$.[1] A *basic semilinear set* in M^n is a set of the form $\{x \in M^n : f_1(x) = \ldots = f_p(x) = 0, g_1(x) > 0, \dots, g_q(x) > 0\}$, where f_i and g_j are linear functions on M^n. Then, (7.6), (7.8) and (7.10) of the above reference say that:

(1) $Th(\mathcal{M})$ admits quantifier elimination and, in particular, the definable subsets of M^n are the *semilinear sets* in M^n, that is, finite unions of basic semilinear sets in M^n.

(2) Every definable function $f : A \subseteq M^n \to M$ is *piecewise linear*, that is, there is a finite partition of A into basic semilinear sets A_i ($i \in \{1, \dots, k\}$), such that $f \restriction_{A_i}$ is linear, for each $i \in \{1, \dots, k\}$.

In fact, the above can be subsumed in a refinement of the classical Cell Decomposition Theorem (henceforth CDT, see [vdD, Chapter 3, (2.11)]) stated below.

[1] We keep the term 'linear' and mean it in the 'affine' sense, conforming to the literature such as [Hud] or [LP].

First, the notion of a 'linear cell' can be defined similarly to the one of a usual cell $\big($[vdD, Chapter 3, (2.2)-(2.4)]$\big)$ by using linear functions in place of definable continuous ones. Namely, for a definable set $X \subseteq M^n$, we let

$$L(X) := \{f : X \to M : f \text{ is linear}\}.$$

If $f \in L(X)$, we denote by $\Gamma(f)$ the graph of f. If $f, g \in L(X) \cup \{\pm\infty\}$ with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and denote by $(f, g)_X$ the 'generalized cylinder' $(f, g)_X = \{(x, y) \in X \times M : f(x) < y < g(x)\}$ between f and g. Then,

- a *linear cell in* M is either a singleton subset of M, or an open interval with endpoinds in $M \cup \{\pm\infty\}$,
- a *linear cell in* M^{n+1} is a set of the form $\Gamma(f)$, for some $f \in L(X)$, or $(f, g)_X$, for some $f, g \in L(X) \cup \{\pm\infty\}$, $f < g$, where X is a linear cell in M^n.

One can then adapt the classical proof of CDT and inductively show:

Linear CDT. *Let $A \subseteq M^n$ and $f : A \to M$ be definable. Then there is a decomposition of M^n that partitions A into finitely many linear cells A_i, such that each $f \restriction_{A_i}$ is linear.* $\big($See [vdD, Chapter 3, (2.10)] for a definition of decomposition of M^n.$\big)$

$D = \langle D, +, \cdot, <, 0, 1\rangle$ is a division ring and $\langle \mathbb{Q}, +, \cdot, <, 0, 1\rangle$ naturally embeds into D. If $a \in M$ and $m \in \mathbb{N}$, we write $\frac{a}{m}$ for $\frac{1}{m}a$, which is also the unique $b \in M$ such that $a = mb = \underbrace{b + \ldots + b}_{m\text{-times}}$, since $\mathcal{M}$ is divisible.

We write $0 := (0, \ldots, 0)$. If $\lambda \in D$, $x = (x_1, \ldots, x_n) \in M^n$ and $X \subseteq M^n$, then $\lambda x := (\lambda x_1, \ldots, \lambda x_n)$ and $\lambda X := \{\lambda x : x \in X\}$, whereas if $\lambda = (\lambda_1, \ldots, \lambda_n) \in D^n$ and $x \in M$, $\lambda x := (\lambda_1 x, \ldots, \lambda_n x)$. If $\lambda \in \mathbb{M}(n, D)$ is an $n \times n$ matrix over D and $x \in M^n$, then λx denotes the resulting n-tuple of the matrix multiplication of λ with x. The unit element of $\mathbb{M}(n, D)$ is denoted by $\mathbb{I}_n$. Again, if $a \in M^n$ and $m \in \mathbb{N}$, then $\frac{a}{m} := \frac{1}{m}a$.

Let $m, n \in \mathbb{N}$. The elements $a_1, \ldots, a_m \in M^n$ are called *linearly independent over* $\mathbb{Z}$ or just $\mathbb{Z}$*-independent* if for all $\lambda_1, \ldots, \lambda_m$ in $\mathbb{Z}$, $\lambda_1 a_1 + \ldots + \lambda_m a_m = 0$ implies $\lambda_1 = \ldots = \lambda_m = 0$. The elements $\lambda_1, \ldots, \lambda_m \in D^n$ are called M*-independent* if for all $t_1, \ldots, t_m \in M$, $\lambda_1 t_1 + \ldots + \lambda_m t_m = 0$ implies $t_1 = \ldots = t_m = 0$.

If $\lambda = (\lambda_1, \ldots, \lambda_n) \in D^n$, then $\lambda^{-1} := (\lambda_1^{-1}, \ldots, \lambda_n^{-1}) \in D^n$.

For $\lambda \in D$, $|\lambda| := \max\{-\lambda, \lambda\}$. For $x \in M$, $|x| := \max\{-x, x\}$, and for $x = (x_1, \ldots, x_n) \in M^n$, $|x| := |x_1| + \ldots + |x_n|$.

Definition 3.1. Let $A \subseteq M^n$.

(i) A is called *convex* if $\forall x, y \in A, \forall q \in \mathbb{Q} \cap [0, 1]$, $qa + (1 - q)b \in A$.

(ii) A is called *bounded* if $\exists r \in M, \forall x \in A$, $|x| \leq r$, that is, $\exists r' \in M$, $A \subseteq \mathcal{B}_0(r')$.

For example, a linear cell is a convex basic semilinear set, and it is bounded if no endpoints or functions involved in its construction are equal to $\pm\infty$. Below we define a special kind of bounded definable convex sets, the 'parallelograms' (Definition 3.5), and make explicit their relation to bounded linear cells (Lemma 3.6).

We consider throughout definable functions $f = (f_1, \ldots, f_n) : M^m \to M^n$, $m, n \in \mathbb{N}$. All definitions apply to f through its components, for example, f is called linear on M^m if every f_i is linear on M^m. Moreover, the Linear CDT holds for definable functions of this form. In fact, a linear function $f : M^n \times M^n \to M^n$ can be written in the usual form, $f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2 + a$, for some fixed $\lambda_i \in \mathbb{M}(n, D)$ and $a \in M^n$.

Definition 3.2. Let $a \in M^n$. We say *a has definable slope* if there are $\lambda \in D^n$ and $e > 0$ in M, such that $a = \lambda e$. In this case, and if $x \in M^n$, we call

$$[0, e] \ni t \mapsto x + \lambda t \in M^n$$

a *linear path from x to $x + a$.*

Remark 3.3. (i) Any two linear paths from x to $x + a$ must have the same image. Indeed, if $a = \lambda_1 e_1 = \lambda_2 e_2$ and $t_1 \in [0, e_1]$, then for $t_2 = \lambda_2^{-1}\lambda_1 t_1 \in [0, e_2]$, we have $\lambda_2 t_2 = \lambda_1 t_1$.

(ii) By Linear CDT, every definable path is, piecewise, a linear path, that is, it is the concatenation of finitely many linear paths.

Lemma 3.4. *Let $A \subseteq M^n$ be definable and convex, and $x, y \in A$. If γ is a linear path from x to y, then* $\mathrm{Im}(\gamma) \subseteq A$.

Proof. Let $\gamma(t) : [0, e] \ni t \mapsto x + \lambda t \in M^n$. Assume, towards a contradiction, that $P := \{t \in [0, e] : x + \lambda t \notin A\} \neq \emptyset$. By o-minimality, P is a finite union of points and open intervals. If it is a finite union of points and t_0 is one of them, then there must be some small $z > 0$ in M such that $t_0 - z, t_0 + z \in [0, e] \setminus P$. But since A is convex, $x + \lambda t_0 = \frac{x+\lambda(t_0-z)+x+\lambda(t_0+z)}{2}$ has to be in A, a contradiction. Similarly, if P contains some intervals, it is possible to find one such with endpoints $t_1 < t_2$, and some $z_1, z_2 \geq 0$ in M, such that $t_1 - z_1, t_2 + z_2 \in [0, e] \setminus P$ and $t_1 < \frac{t_1-z_1+t_2+z_2}{2} < t_2$. Then $x + \lambda\frac{t_1-z_1+t_2+z_2}{2} = \frac{x+\lambda(t_1-z_1)+x+\lambda(t_2+z_2)}{2} \in A$, again a contradiction. □

Definition 3.5. Let $a_1, \ldots, a_m \in M^n$, $0 < m \leq n$, have definable slopes, and $a \in M^n$. Then, the *closed m-parallelogram anchored at a and generated by $a_1, \ldots, a_m$*, denoted by $\overline{P}_a(a_1, \ldots, a_m)$, is the closed definable set

$$a + \{\lambda_1 t_1 + \ldots + \lambda_m t_m : t_i \in [0, e_i]\},$$

where $a_i = \lambda_i e_i$, $e_i > 0$, $1 \leq i \leq m$. The *open m-parallelogram anchored at a and generated by $a_1, \ldots, a_m$*, denoted by $P_a(a_1, \ldots, a_m)$, is the definable set

$$a + \{\lambda_1 t_1 + \ldots + \lambda_m t_m : t_i \in (0, e_i)\}.$$

We just say *open (*or *closed) m-parallelogram* if a and $a_1, \ldots, a_m$ are not specified. The 2^m elements $a + \lambda_1 t_1 + \ldots + \lambda_m t_m$, $t_i = 0, e_i$, are called the *corners* of $\overline{P}_a(a_1, \ldots, a_m)$ and $P_a(a_1, \ldots, a_m)$; the element $\frac{1}{2^m} \sum\limits_{t_i = 0, e_i} \big(a + \lambda_1 t_1 + \ldots + \lambda_m t_m\big) = a + \frac{1}{2}\sum_{i=1}^m a_i$ is called their *center.*

Remark 3.3(i) guarantees that the definition of $\overline{P}_a(a_1, \ldots, a_m)$ and $P_a(a_1, \ldots, a_m)$ does not depend on the choice of λ_i and e_i, $1 \leq i \leq m$.

Clearly, an open or closed m-parallelogram is a definable bounded convex set.

Lemma 3.6. *The closure of every bounded n-dimensional linear cell $Y \subset M^n$, $n > 0$, is a finite union of closed n-parallelograms.*

Proof. By induction on n.

$\mathbf{n = 1}$. $Y = (a, b) \subset M$, $a, b \in M$. Then $\overline{Y} = \overline{P}_a(b - a)$.

$\mathbf{n > 1}$. A bounded n-dimensional linear cell Y must have the form $Y = (f, g)_X$, for some $(n-1)$-dimensional linear cell X in M^{n-1} and $f < g \in L(X)$. By Inductive

Hypothesis, $\overline{X}$ is a finite union of closed $(n-1)$-parallelograms, and thus it suffices to show that for any closed $(n-1)$-parallelogram $Q \subset M^{n-1}$ and $f < g \in L(Q)$, $\overline{(f,g)_Q}$ is a finite union of closed n-parallelograms. Let $Q = \overline{P}_{q_0}(q_1,\dots,q_{n-1})$ in M^{n-1}, $a_0 = \big(q_0, f(q_0)\big)$, $b_0 = \big(q_0, g(q_0)\big)$, and $\forall i \in \{0,\dots,n-1\}$,

$$a_i = \big(q_0+q_i, f(q_0+q_i)\big) - a_0 = \big(q_i, f(q_0+q_i) - f(q_0)\big) \in M^n$$

and

$$b_i = \big(q_0+q_i, g(q_0+q_i)\big) - b_0 = \big(q_i, g(q_0+q_i) - g(q_0)\big) \in M^n.$$

Then, $\Gamma(f) = \overline{P}_{a_0}(a_1,\dots,a_{n-1})$ and $\Gamma(g) = \overline{P}_{b_0}(b_1,\dots,b_{n-1})$. Indeed, it is not very hard to see that for $0 < i \le n-1$, if $[0,e_i] \ni t_i \mapsto q_i(t_i) \in M^{n-1}$ is a linear path from 0 to q_i, then

$$[0,e_i] \ni t_i \mapsto a_i(t_i) := \Big(q_i(t_i), f\big(q_0+q_i(t_i)\big) - f(q_0)\Big) \in M^n$$

is a linear path from 0 to a_i, and

$$[0,e_i] \ni t_i \mapsto b_i(t_i) := \Big(q_i(t_i), g\big(q_0+q_i(t_i)\big) - g(q_0)\Big) \in M^n$$

is a linear path from 0 to b_i. Moreover, for any $x = q_0 + \sum_{i=1}^{n-1} q_i(t_i) \in Q$, we have $f(x) = \sum_{i=1}^{n-1} f\big(q_0+q_i(t_i)\big) - (n-2)f(q_0)$, since by linearity of f, for any $j \in \{2,\dots,n-1\}$, $f\left(q_0 + \sum_{i=1}^{j} q_i(t_i)\right) - f\left(q_0 + \sum_{i=1}^{j-1} q_i(t_i)\right) = f\big(q_0+q_j(t_j)\big) - f(q_0)$. Thus,

$$\begin{aligned} a_0 + \sum_{i=1}^{n-1} a_i(t_i) &= \big(q_0, f(q_0)\big) + \sum_{i=1}^{n-1}\Big(q_i(t_i), f\big(q_0+q_i(t_i)\big) - f(q_0)\Big) \\ &= \left(q_0 + \sum_{i=1}^{n-1} q_i(t_i), \sum_{i=1}^{n-1} f\big(q_0+q_i(t_i)\big) - (n-2)f(q_0)\right) = \big(x, f(x)\big). \end{aligned}$$

It follows that $\Gamma(f) = \overline{P}_{a_0}(a_1,\dots,a_{n-1})$. Similarly, $\Gamma(g) = \overline{P}_{b_0}(b_1,\dots,b_{n-1})$.

Now, if $\exists c \in M^n, \forall i \in \{0,\dots,n-1\}$, $b_i - a_i = c$, then for all $i > 0$, $a_i - a_0 = b_i - b_0$ and $\overline{(f,g)_Q}$ is the closed n-parallelogram $\overline{P}_{a_0}(a_1,\dots,a_{n-1}, b_0-a_0)$. Indeed, one first can see that $\forall x \in Q$, $g(x) - f(x) = b_0 - a_0 = c$, and thus $\overline{(f,g)_Q} = \{(x,y) \in M^{n-1} \times M : x \in Q,\ y \in f(x) + [0, (b_0)_n - (a_0)_n]\}$. On the other hand, consider the linear path $[0, b_0 - a_0] \ni t \mapsto (b_0-a_0)(t) := (0,t) \in M^{n-1} \times M$ from 0 to $(0, b_0 - a_0)$ in M^n. Then, every element in $\overline{P}_{a_0}(a_1,\dots,a_{n-1}, b_0-a_0)$ has the form $a_0 + \sum_{i=1}^{n-1} a_i(t_i) + (b_0-a_0)(t) = \big(x, f(x)\big) + (0,t) = \big(x, f(x)+t\big)$, for $x \in Q$ and $t \in [0, (b_0)_n - (a_0)_n]$.

Otherwise, we may assume that $\overline{(f,g)_Q}$ is such that for some $i \in \{0,\dots,n-1\}$, $a_i = b_i$. Indeed, let $C = \{|b_i - a_i| : 0 \le i \le n-1\}$, and let $j \in \{0,\dots,n-1\}$ be such that $|b_j - a_j| = (b_j)_n - (a_j)_n$ is minimum in C. If, say, $j = 0$, and $a_0 \ne b_0$, it is easy to see as before that $\overline{(f,g)_Q} = \overline{(f,f')_Q} \cup \overline{P}_{a_0}(b_1,\dots,b_{n-1}, b_0-a_0)$, where $\forall x \in Q, f'(x) = g(x) - (b_0 - a_0)$, that is, $\overline{(f,g)_Q}$ is the union of the closure of a cell of the desired form and a closed n-parallelogram.

We can further assume that all corners of $\Gamma(f)$ and $\Gamma(g)$ but one coincide. For, if $\Gamma(f) = \overline{P}_{a_0}(a_1,\dots,a_{n-1})$ and $\Gamma(g) = \overline{P}_{a_0}(b_1,\dots,b_{n-1})$, with say $a_1 \ne b_1$ and $a_2 \ne b_2$, then $\overline{(f,g)_Q} = \overline{(f,f')_Q} \cup \overline{(f',g)_Q}$, where $f' \in L(Q)$ such that $\Gamma(f') = \overline{P}_{a_0}(b_1, a_2,\dots,a_{n-1})$. Clearly, the corners of $\Gamma(f)$ and $\Gamma(f')$ differ by one, and the corners of $\Gamma(f')$ and $\Gamma(g)$ differ by one less than those of $\Gamma(f)$ and $\Gamma(g)$. Thus, repeating this process, we see that $\overline{(f,g)_Q}$ is a union of closures of cells of the desired form.

Now let $\Gamma(f) = \overline{P}_{a_0}(a_1, a_2, \ldots, a_{n-1})$ and $\Gamma(g) = \overline{P}_{a_0}(b_1, a_2, \ldots, a_{n-1})$. Let $\bar{a} := (a_2, \ldots, a_{n-1})$. Then, $\overline{(f,g)_Q} = P_1 \cup P_2 \cup P_3$, where

$P_1 = \overline{P}_{a_0}\left(\frac{a_1}{2}, \frac{b_1}{2}, \bar{a}\right)$,

$P_2 = \overline{P}_{a_0+\frac{a_1}{2}}\left(\frac{a_1}{2}, \frac{b_1-a_1}{2}, \bar{a}\right)$, and

$P_3 = \overline{P}_{a_0+\frac{b_1}{2}}\left(\frac{b_1}{2}, \frac{a_1-b_1}{2}, \bar{a}\right)$.

Indeed, let $x = q_0 + \sum_{i=1}^{n-1} q_i(t_i) \in Q$, and $\big(x, f(x)+t\big) \in \overline{(f,g)_Q}$, $t \in [0, g(x)-f(x)]$. Then the following are easy to check. If $t_1 \leq \frac{e_1}{2}$, then $(x, f(x)+t) \in P_1$. If $t_1 \geq \frac{e_1}{2}$, then if $t \leq \frac{(b_1)_n-(a_1)_n}{2}$, $\big(x, f(x)+t\big) \in P_2$, whereas if $t \geq \frac{(b_1)_n-(a_1)_n}{2}$, $\big(x, f(x)+t\big) \in P_3$. □

For the rest of this section, let $G = \langle G, \oplus, e_G\rangle$ be a definable group with $G \subseteq M^n$ and $\dim(G) = m \leq n$.

Note that if a definable set $A \subseteq M^n$ is unbounded, then there is a definable continuous embedding $\gamma : [0, \infty) \to A$.

Lemma 3.7. *If G is definably compact, then G is definably bijective to a bounded subset of M^m. Thus, in this case, we can assume $m = n$ (see Remark 2.2).*

Proof. Recall, G admits a finite t-open covering $\{S_i\}_{i\in J}$, such that each S_i is definably homeomorphic to an open subset K_i of M^m via $\phi_i : S_i \to K_i$. It is not hard to see that it suffices to show that each K_i is bounded in M^m. If, say, K_1 is not, there must be a definable continuous embedding $\gamma : [0, \infty) \to K_1$. Since G is definably compact, there is some $g \in G$ with $\lim^t_{x\to\infty} \phi_1^{-1}\big(\gamma(x)\big) = g$. If $g \in S_l$, $l \in J$, take a bounded open subset B of K_l in M^m containing $\phi_l(g)$. Then the restriction of the map $\phi_l \circ \phi_1^{-1} \circ \gamma$ on some $[a, \infty)$ such that $\phi_l \circ \phi_1^{-1} \circ \gamma\big([a, \infty)\big) \subseteq B$ is a piecewise linear bijection between a bounded and an unbounded set in M^m, a contradiction. □

Definition 3.8. Assume G is abelian. Let $X \subseteq G \subseteq M^n$. A *$\oplus$-translate of X* is a set of the form $a \oplus X$, for $a \in G$. We say that X *is generic (in G)* if finitely many $\oplus$-translates of X cover G.

Fact 3.9. *Assume G is abelian. Then,*
(i) Every large definable subset of G is generic.
Assume, further, that X is a definable subset of G. Then,
(ii) If $X \subseteq G$ is generic, then $\dim(X) = \dim(G)$.
(iii) $X \subseteq G$ is generic if and only if $\overline{X}^t$ is generic.

Proof. (i) is by [Pi1], whereas (ii) and (iii) constitute [PePi, Lemma 3.4]. □

Let us note here that, although in [PePi] the authors work over an o-minimal expansion $\mathcal{M}$ of a real closed field, their proofs of several facts about generic sets, such as [PePi, Lemma 3.4], that is, Fact 3.9 above, go through in the present context as well. More significantly, their Corollary 3.9 holds. To spell out a few more details, their use of the field structure of $\mathcal{M}$ is to ensure that G is affine $\big($[vdD, Chapter 10, (1.8)]$\big)$, and, therefore, that a definably compact subset X of G is closed and bounded $\big($[PeS]$\big)$. Theorem 2.1 from [PePi] (which is extracted from Dolich's work, and is shown in their Appendix to be true if $\mathcal{M}$ expands an ordered group), then applies and shows their Lemma 3.6 and, following, Corollary 3.9. Although in our context G may not be affine, [PePi, Theorem 2.1] can be restated for any $X \subseteq G$, which is definably compact, instead of closed and bounded, assuming G is definably compact, as below. The rest of the proof of [PePi, Corollary 3.9] then works identically.

Lemma 3.10. *Let both G and $X \subseteq G$ be definably compact, and $\mathcal{M}_0$ a small elementary substructure of $\mathcal{M}$ (that is, $|M_0| < |M|$), such that the manifold structure of G is $\mathcal{M}_0$-definable. Then the following are equivalent:*
(i) The set of $\mathcal{M}_0$-conjugates of X is finitely consistent.
(ii) X has a point in $\mathcal{M}_0$.
Therefore ([PePi, Corollary 3.9]), *if G is abelian, the union of any two non-generic definable subsets of G is also non-generic.*

Proof. First, G is Hausdorff, since M^m is and G is locally homeomorphic to M^m. One can then show that there are $\mathcal{M}_0$-definable t-open subsets $O_i \subseteq G$, $i \in J$, such that $G = \bigcup_{i\in J} O_i$ and $\overline{O_i}^t \subset S_i$ (see [BO1, Lemmas 10.4, 10.5], for example, where the authors work over a real closed field but their arguments go word-by-word through in the present context, as well). Now, for the non-trivial direction $(i) \Rightarrow (ii)$, let $X \subseteq G$, $X = \bigcup_{i\in J} X_i$, with $X_i := X \cap \overline{O_i}^t$, and assume that the set of $\mathcal{M}_0$-conjugates of X is finitely consistent. Since O_i and the chart maps $\phi_i : S_i \to M^m$ are $\mathcal{M}_0$-definable, if $f \in \mathrm{Aut}_{\mathcal{M}_0}(M)$, then $f(X_i) \subseteq \overline{O_i}^t$, and thus the set $\big\{ \bigcup_{i\in J} \phi_i\big(f(X_i)\big)\big\}_{f\in \mathrm{Aut}_{\mathcal{M}_0}(M)}$ is finitely consistent. Moreover, it is not hard to see that $f\big(\bigcup_{i\in J}\phi_i(X_i)\big) = \bigcup_{i\in J}\phi_i\big(f(X_i)\big)$, which gives that the set of $\mathcal{M}_0$-conjugates of $\bigcup_{i\in J}\phi_i(X_i)$ is finitely consistent. Since each X_i is definably compact, $\bigcup_{i\in J}\phi_i(X_i)$ is closed and bounded in M^m. By [PePi, Theorem 2.1], $\bigcup_{i\in J}\phi_i(X_i)$ has a point in $\mathcal{M}_0$, say $a \in \phi_1(X_1)$, and thus X_1 has a point b in $\mathcal{M}_0$ (since $\mathcal{M}_0 \prec \mathcal{M} \models \exists y \in X_1 \phi_1(y) = a$). □

Remark 3.11. The proof (and the result) of Lemma 3.10 are valid in any o-minimal expansion $\mathcal{M}$ of an ordered group. Moreover, the proof of Lemma 3.10 shows that Lemma 3.7 is also valid in any o-minimal expansion $\mathcal{M}$ of an ordered group. Indeed, with the above notation, each $\overline{O_i}^t$ is definably compact (as a t-closed subset of the definably compact G), hence $\phi_i(\overline{O_i}^t) \subseteq M^m$ is definably compact in M^m and thus (closed and) bounded.

4. The proof of Theorem 1.4

Outline. We split our proof into three steps. We let $G = \langle G, \oplus, e_G\rangle$ be a $\emptyset$-definable group with $G \subseteq M^n$.

In Step I, we begin with a local analysis on G and show that the set V^G (from Definition 2.4) is large in G. We then let G be n-dimensional, definably compact and t-connected, and, based on the set V^G, we compare the two group operations $\oplus$ and $+$. A key notion is that of a 'jump' of a t-path (Definition 4.16), and the main results of this first step are Lemma 4.23 and Proposition 4.24.

In Step II, we invoke [PePi, Corollary 3.9] (see Lemma 3.10 here) in order to establish the existence of a generic open n-parallelogram H in G, which is used to generate a subgroup $U \leqslant M^n$. Using Lemma 4.23(i) from Step I, we can define a group homomorphism ϕ from U onto G, and let $L := \ker(\phi)$.

In Step III, we use Proposition 4.24 to prove that L is a lattice generated by some elements of M^n recovered in Step I, namely, by some $\mathbb{Z}$-linear combinations of 'jump vectors'. Then we use H to obtain a 'standard part' map from U to $\mathbb{R}^n$. This allows us to compute the rank of L and finish the proof.

STEP I. Comparing $\oplus$ with $+$. We let $G = \langle G, \oplus, e_G \rangle$ be a $\emptyset$-definable group with $G \subseteq M^n$ and $\dim(G) = m \le n$. (We do not yet assume that G is definably compact or t-connected.) Our first goal is to show that V^G is a large subset of G, which among other things implies that G is locally isomorphic to $M^m = \langle M^m, +, 0 \rangle$.

A consequence of the Linear CDT is that for any two independent dim-generic elements a and b of G, there are t-neighborhoods V_a of a and V_b of b in G, such that for all $x \in V_a$ and $y \in V_b$, $x \oplus y = \lambda x + \mu y + d$, for some fixed $\lambda, \mu \in \mathbb{M}(n, D)$, and $d \in M^n$. Moreover, λ and μ have to be invertible matrices (for example, setting $y = b$, $x \oplus b = \lambda x + \mu b + d$ is invertible, showing that λ is invertible).

Proposition 4.1. *For every dim-generic element a of G, there exists a t-neighborhood V_a of a in G, such that for all $x, y \in V_a$,*

$$x \ominus a \oplus y = x - a + y.$$

Proof. We proceed through several lemmas.

Lemma 4.2. *For every two independent dim-generics $a, b \in G$, there exist t-neighborhoods V_a of a and V_b of b in G, invertible λ, $\lambda' \in \mathbb{M}(n, D)$, and $c = b - \lambda a$, $c' = b - \lambda' a \in M^n$, such that for all $x \in V_a$,*

$$x \ominus a \oplus b = \lambda x + c \in V_b \quad \text{and} \quad \ominus a \oplus b \oplus x = \lambda' x + c' \in V_b.$$

Proof. Since a and b are independent dim-generics of G, a and $\ominus a \oplus b$ are independent dim-generics of G as well. Therefore, there are t-neighborhoods V_a of a and $V_{\ominus a \oplus b}$ of $\ominus a \oplus b$ in G, as well as invertible $\lambda, \mu \in \mathbb{M}(n, D)$ and $d \in M^n$, such that $\forall x \in V_a$, $\forall y \in V_{\ominus a \oplus b}$, $x \oplus y = \lambda x + \mu y + d$. In particular, for all $x \in V_a$, $x \ominus a \oplus b = \lambda x + \mu(\ominus a \oplus b) + d$. Letting $c = \mu(\ominus a \oplus b) + d$ and $V_b = \{x \ominus a \oplus b : x \in V_a\}$ shows the first equality. That $c = b - \lambda a$, it can be verified by setting $x = a$. The second equality can be shown similarly. $\square$

Lemma 4.3. *Let a be a dim-generic element of G. Then there exist a t-neighborhood V_a of a in G, $\lambda, \mu \in \mathbb{M}(n, D)$ and $d \in M^n$, such that for all $x, y \in V_a$,*

$$x \ominus a \oplus y = \lambda x + \mu y + d.$$

Proof. Take a dim-generic element a_1 of G independent from a. Then $a_2 = a \ominus a_1$ is also a dim-generic element of G independent from a. By Lemma 4.2, we can find t-neighborhoods V_{a_1}, V_{a_2}, V_a of a_1, a_2, a, respectively, in G, as well as $\lambda_1, \lambda_2 \in \mathbb{M}(n, D)$ and $c_1, c_2 \in M^n$, such that $\forall x \in V_a$, $x \ominus a \oplus a_1 = \lambda_1 x + c_1 \in V_{a_1}$ and for all $y \in V_a$, $\ominus a \oplus a_2 \oplus y = \lambda_2 y + c_2 \in V_{a_2}$. Moreover, since a_1 and $a_2 = a \ominus a_1$ are independent dim-generics of G, we could choose V_{a_1}, V_{a_2} and V_a be such that for some fixed $\nu, \xi \in \mathbb{M}(n, D)$ and $o \in M^n$, we have: $\forall x \in V_{a_1}$, $\forall y \in V_{a_2}$, $x \oplus y = \nu x + \xi y + \varepsilon$. Now for all $x, y \in V_a$, we have:

$$\begin{aligned} x \ominus a \oplus y &= x \ominus a \oplus a_1 \ominus a_1 \oplus y \\ &= (x \ominus a \oplus a_1) \oplus (\ominus a \oplus a_2 \oplus y) \\ &= \nu(\lambda_1 x + c_1) + \xi(\lambda_2 y + c_2) + o \\ &= \nu\lambda_1 x + \xi\lambda_2 y + \nu c_1 + \xi c_2 + o \end{aligned}$$

Setting $\lambda = \nu\lambda_1, \mu = \xi\lambda_2$, and $d = \nu c_1 + \xi c_2 + o$ finishes the proof of the lemma. □

We can now finish the proof of Proposition 4.1. By Lemma 4.3, there exists a t-neighborhood V_a of a in G, $\lambda, \mu \in \mathbb{M}(n, D)$ and $d \in M^n$, such that for all $x, y \in V_a$, $x \ominus a \oplus y = \lambda x + \mu y + d$. In particular, for all $x, y \in V_a$,

$$y = a \ominus a \oplus y = \lambda a + \mu y + d$$

$$x = x \ominus a \oplus a = \lambda x + \mu a + d$$

and, therefore, $x + y = (\lambda x + \mu y + d) + (\lambda a + \mu a + d)$. But, $\lambda x + \mu y + d = x \ominus a \oplus y$, and

$$a = a \ominus a \oplus a = \lambda a + \mu a + d.$$

Hence, $x + y = (x \ominus a \oplus y) + a$, or, $x \ominus a \oplus y = x - a + y$. □

Corollary 4.4. *G is 'definably locally isomorphic' to M^m. That is, there is a definable homeomorphism f from some t-neighborhood V_{e_G} of e_G in G to a definable open neighborhood W_0 of 0 in M^m, such that:*

(i) for all $x, y \in V_{e_G}$, if $x \oplus y \in V_{e_G}$, then $f(x \oplus y) = f(x) + f(y)$, and

(ii) for all $x, y \in W_0$, if $x + y \in W_0$, then $f^{-1}(x + y) = f^{-1}(x) \oplus f^{-1}(y)$.

(See [Pon, Definition 30] for more on this definition of a local isomorphism.)

Proof. Let a be a dim-generic element of G. The function $G \ni x \mapsto x \oplus a \in G$ witnesses that the topological group $(G, \oplus, e_G)$ is definably isomorphic to $(G, *, a)$, where $x * y = x \ominus a \oplus y$ (Remark 2.2). Now, since a is dim-generic, some t-neighborhood V_a of a in G can be projected homeomorphically onto an open subset W_a of M^m, inducing on W_a the group structure from V_a. We can thus assume that $V_a \subseteq M^m$. By Proposition 4.1, the definable function $f : G \ni x \mapsto x - a \in M^m$ witnesses, easily, that $(G, *, a)$ is definably locally isomorphic to M^m. Thus, $(G, \oplus, e_G)$ is (definably isomorphic to a group which is) definably locally isomorphic to M^m. □

The following corollary is already known; for example, see [Ed1, Corollary 6.3] or [PeSt, Corollary 5.1]. It can also be extracted from [LP].

Corollary 4.5. *G is abelian-by-finite.*

Proof. Let V_{e_G} be as in Corollary 4.4. Since $\oplus$ is t-continuous, there is a t-open $U' \subseteq G$ with $\forall x, y \in U'$, $x \oplus y \in V_{e_G}$. Thus, if we let $U := U' \cap V_{e_G}$, then $\forall x, y \in U$, $x \oplus y = f^{-1}\big(f(x) + f(y)\big) = f^{-1}\big(f(y) + f(x)\big) = y \oplus x$.

Now let G^0 be the t-connected component of e_G in G. Then for every element $a \in U$, its centralizer $C(a) = \{x \in G : a \oplus x = x \oplus a\}$ contains the t-open (m-dimensional) subset $U \subseteq G$, and thus $G^0 \subseteq C(a)$. It follows that the center $Z(G^0) = \{x \in G^0 : \forall y \in G^0, x \oplus y = y \oplus x\}$ of G^0 contains U, thus $Z(G^0)$ must have dimension m and be equal to G^0. That is, G^0 is abelian. □

We fix for the rest of the paper a definable group $G = \langle G, \oplus, e_G \rangle$, definably compact and t-connected, with $G \subseteq M^n$. By Lemma 3.7, we assume $\dim(G) = n$. By Corollary 4.5, G is abelian.

Proposition 4.1 says that the set V^G is large in G. We omit the index 'G' and write just V. Then, V is t-open as well as open, and, by cell decomposition, it is the disjoint union of finitely many open definably connected components $V_0, \dots, V_N$, that is, $V = \bigsqcup_{i \in I} V_i$, for a fixed index set $I := \{0, \dots, N\}$.

Next goal is to show that the property

$$(u+\varepsilon)\ominus u=(v+\varepsilon)\ominus v$$

can be assumed to be true for any $u, v \in V$ and 'small' $\varepsilon \in M^n$ (Corollary 4.12). In what follows, whenever we write a property that includes an expression of the form '$x \oplus y$', it is meant that $x, y \in G$ (and the property holds).

Corollary 4.6. *For all $u \in V$, there is $r > 0$ in M, such that for all $v \in \mathcal{B}_u(r)$ and $\varepsilon \in (-r, r)^n$,*

$$(u+\varepsilon)\ominus u=(v+\varepsilon)\ominus v.$$

Proof. By definition of V, there is $r > 0$ in M, such that for all $v \in \mathcal{B}_u(r)$ and $\varepsilon \in (-r, r)^n$,

$$(u+\varepsilon)\ominus u\oplus v=u+\varepsilon-u+v=v+\varepsilon.$$

□

Lemma 4.7. *For all u, v in the same definably connected component of V, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, +r)^n$,*

$$(u+\varepsilon)\ominus u=(v+\varepsilon)\ominus v.$$

Proof. Let V_i be a definably connected component of V and u some element in V_i. We show that the set

$$\Gamma=\{v\in V_i : \exists r>0\in M\,\forall\varepsilon\in(-r,+r)^n\,[(u+\varepsilon)\ominus u=(v+\varepsilon)\ominus v]\}$$

is a nonempty clopen subset of V_i. First, Γ is nonempty since it contains u. To show that Γ is open, consider an element $v \in \Gamma$. Let $r_v \in M$ be such that $\forall\varepsilon \in (-r_v, r_v)^n$, $(u+\varepsilon)\ominus u = (v+\varepsilon)\ominus v$. By Corollary 4.6, there is $s_v > 0$ in M such that for all $v' \in \mathcal{B}_v(s_v)$ and $\varepsilon \in (-s_v, s_v)^n$, $(v+\varepsilon)\ominus v = (v'+\varepsilon)\ominus v'$. By letting $r := \min\{r_v, s_v\}$, we obtain that for all $v' \in \mathcal{B}_v(r)$, for all $\varepsilon \in (-r, r)^n$,

$$(v'+\varepsilon)\ominus v'=(v+\varepsilon)\ominus v=(u+\varepsilon)\ominus u,$$

that is, $\mathcal{B}_r(v) \subseteq \Gamma$, and therefore Γ is open.

To show that Γ is closed in V_i, pick some v in $V_i \setminus \Gamma$. It should satisfy

$$\forall r>0\,\exists\varepsilon_v\in(-r,r)^n\,[(u+\varepsilon)\ominus u\neq(v+\varepsilon)\ominus v]. \tag{1}$$

Now, let as before $s_v > 0$ be so that for all $v' \in \mathcal{B}_v(s_v)$ and $\varepsilon \in (-s_v, s_v)^n$, $(v+\varepsilon)\ominus v = (v'+\varepsilon)\ominus v'$. We want to show $v' \in V \setminus \Gamma$, that is, $\forall r_{v'} > 0$,

$$\exists\varepsilon_{v'}\in(-r_{v'},r_{v'})^n[(u+\varepsilon_{v'})\ominus u\neq(v'+\varepsilon_{v'})\ominus v']. \tag{2}$$

It suffices to show (2) for all $r_{v'}$ with $\mathcal{B}_{v'}(r_{v'}) \subseteq \mathcal{B}_v(s_v)$. Let $r_{v'}$ be one such. Apply (1) for $r_{v'}$ to get an $\varepsilon_{v'} \in (-r_{v'}, r_{v'})^n \subseteq (-s_v, s_v)^n$ satisfying $(u+\varepsilon_{v'})\ominus u \neq (v+\varepsilon_{v'})\ominus v$. But since $\varepsilon_{v'} \in (-s_v, s_v)^n$, we also have $(v+\varepsilon_{v'})\ominus v = (v'+\varepsilon_{v'})\ominus v'$. It follows that $(u+\varepsilon_{v'})\ominus u \neq (v'+\varepsilon_{v'})\ominus v'$. □

More generally, the following is true.

Lemma 4.8. *There are invertible $\lambda_0, \dots, \lambda_N \in \mathbb{M}(n, D)$ such that for any $i, j \in I = \{0, \dots, N\}$, $u \in V_i$ and $v \in V_j$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$,*

$$(u+\lambda_i\varepsilon)\ominus u=(v+\lambda_j\varepsilon)\ominus v.$$

In particular, $\lambda_0 = \mathbb{I}_n$.

Proof. By Lemma 4.2, for any two independent dim-generics $u \in V_0$ and $v \in V_j$, $j \in I$, there is invertible $\lambda_j \in \mathbb{M}(n, D)$ such that for all x in some small t-neighborhood of u in G, $x \ominus u \oplus v = \lambda_j x + v - \lambda_j u$, or, equivalently, for sufficiently small ε, $(u+\varepsilon) \ominus u \oplus v = \lambda_j(u+\varepsilon) + v - \lambda_j u = v + \lambda_j \varepsilon$, that is, $(u+\varepsilon) \ominus u = (v+\lambda_j\varepsilon) \ominus v$. By Lemma 4.7, the last equation holds for any $u \in V_0$ and $v \in V_j$, perhaps for some smaller epsilon's. Clearly, $\lambda_0 = \mathbb{I}_n$. Now, pick any $i, j \in I$, and any $v_0 \in V_0$, $u \in V_i$, $v \in V_j$. We derive that for sufficiently small ε:

$$(u + \lambda_i \varepsilon) \ominus u = (v_0 + \varepsilon) \ominus v_0 = (v + \lambda_j \varepsilon) \ominus v.$$

□

We next show (Lemma 4.11) that all λ_i's in Lemma 4.8 can be assumed to be equal to $\mathbb{I}_n$. First, let us notice it is harmless to assume $0 = e_G \in V$, which in particular means that in a t-neighborhood of 0 the $\mathcal{M}$- and t- topologies coincide.

Lemma 4.9. $(G, \oplus, e_G)$ *is definably isomorphic to a topological group* $(G', +_1, 0)$ *with* $0 \in V^{G'}$.

Proof. Pick a dim-generic point $b \in G$. Consider the definable bijection

$$f : G \ni x \mapsto (x \oplus b) - b \in f(G) \subseteq M^n.$$

Let $G' := f(G)$ and let $\langle G', +_1, 0 = f(e_G)\rangle$ be the induced topological group structure on G' by f. Then f is a definable isomorphism between $\langle G, \oplus, e_G\rangle$ and $\langle G', +_1, 0 = f(e_G)\rangle$ (Remark 2.2). We show that

$$V^{G'} = V - b,$$

and, therefore, since $b \in V$, we have $0 \in V^{G'}$.

For all $x, y, c \in G'$, we have that $x + b, y + b, c + b \in G$ and the following holds:

$$\begin{aligned} x -_1 c +_1 y &= f\big(f^{-1}(x) \ominus f^{-1}(c) \oplus f^{-1}(y)\big) \\ &= \big([(x+b) \ominus b \ominus (c+b) \oplus b \oplus (y+b) \ominus b] \oplus b\big) - b \\ &= [(x+b) \ominus (c+b) \oplus (y+b)] - b. \end{aligned}$$

Now, assume that $c+b \in V$. We claim that $c \in V^G$. Indeed, if x, y are sufficiently close to c, then $x + b$, $y + b$ will be close to $c + b \in V$, hence

$$[(x+b) \ominus (c+b) \oplus (y+b)] - b = x + b - c - b + y + b - b = x - c + y.$$

This shows $V^{G'} \subseteq V - b$ (which is what we need). The inverse inclusion can be shown similarly. □

Remark 4.10. The above proof can be split into two parts: (i) for every element b in G, the definable bijection $f_1 : G \ni x \mapsto x \oplus b \in G$ preserves V, and (ii) for every element b in G, the definable bijection $f_2 : G \ni x \mapsto x - b \in G'$ maps V to $V^{G'}$, that is, $V^{G'} = V - b$. Later, we use the property that a bijection such as f_2 maps m-parallelograms to m-parallelograms.

We let V_0 be the component of V that contains $0 = e_G$.

Lemma 4.11. G *is definably isomorphic to a group* $G' = \langle G', +_1, 0\rangle$ *whose corresponding* $\lambda_i^{G'}$*'s (as in Lemma 4.8) are all equal to* $\mathbb{I}_n$.

Proof. For any $i \in I$, let a_i be some element in V_i. Consider the definable function $f : G \to M^n$, such that

$$f(x) = \begin{cases} \lambda_i^{-1}(x - a_i) + a_i & \text{if } x \in V_i, \\ x & \text{if } x \in G \setminus V. \end{cases}$$

We can assume that f is one-to-one, by definably moving the definably connected components of G sufficiently 'far away' from each other if needed, which is possible, by Lemma 3.7. We show that in the induced group $G' = \langle f(G) = G', +_1, f(0) = 0\rangle$ the corresponding set $V^{G'}$ is exactly the set $f(V) = f(V_0) \bigsqcup \ldots \bigsqcup f(V_N)$, with $f(V_0), \ldots, f(V_N)$ as its definably connected components. First, notice that for $x \in V_i \subseteq G$ and ε 'small', $\lambda_i\varepsilon$ is also small, and $f(x + \lambda_i\varepsilon) = \lambda_i^{-1}(x + \lambda_i\varepsilon - a_i) + a_i = \lambda_i{}^{-1}(x - a_i) + a_i + \varepsilon = f(x) + \varepsilon$. Thus, for all $x, y, c \in G'$, with x, y close to c, $f^{-1}(x), f^{-1}(y)$ must be close to $f^{-1}(c)$. Moreover, if $f^{-1}(c) \in V_i$, then $x, y, c \in f(V_i)$ and:

$$\begin{aligned} x -_1 c +_1 y = \ & f\big(f^{-1}(x) \ominus f^{-1}(c) \oplus f^{-1}(y)\big) = f\big(f^{-1}(x) - f^{-1}(c) + f^{-1}(y)\big) \\ = \ & \lambda_i{}^{-1}\Big(\big[\big(\lambda_i(x - a_i) + a_i\big) - \big(\lambda_i(c - a_i) + a_i\big) + \big(\lambda_i(y - a_i) + a_i\big)\big] - a_i\Big) + a_i \\ = \ & x - c + y, \end{aligned}$$

This shows that $f(V_i) \subseteq V_i^{G'}$. Similarly for the inverse inclusion.

It then suffices to show that for any $i \in \{0, \ldots, N\}$, for all $u = f(u) \in V_0^{G'} = V_0$, $f(v) \in V_i^{G'}$, and sufficiently small ε,

$$(u + \varepsilon) -_1 u = \big(f(v) + \varepsilon\big) -_1 f(v).$$

We have,

$$\big(f(v)+\varepsilon\big)-_1 f(v) = f(v+\lambda_i\varepsilon)-_1 f(v) = f\big((v+\lambda_i\varepsilon)\ominus v\big) = f\big((u+\varepsilon)\ominus u\big) = (u+\varepsilon)-_1 u,$$

by Lemma 4.8 and since f is the identity on V_0. □

By Proposition, we can assume that for any $i \in I = \{0, \ldots, N\}$, $\lambda_i = \mathbb{I}_n$. Therefore, Lemma 4.8 becomes:

Corollary 4.12. *For all $u, v \in V$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$,*

$$(u + \varepsilon) \ominus u = (v + \varepsilon) \ominus v.$$

Corollary 4.13. *For all $u \in V$, $v \in G$, such that $u \oplus v \in V$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$,*

$$(u + \varepsilon) \oplus v = (u \oplus v) + \varepsilon. \tag{3}$$

Proof. By Corollary 4.12, there is $r > 0$ in M, such that $\forall \varepsilon \in (-r, r)^n$,

$$(u + \varepsilon) \ominus u = [(u \oplus v) + \varepsilon] \ominus (u \oplus v).$$

□

The final goal in this first step (Lemma 4.23 and Proposition 4.24) is to obtain suitable versions of the equation (3), where i) u, v and $u \oplus v$ are in G, and ii) ε is arbitrary in M^n.

Definition 4.14. We let $\sim_G$ be the following definable equivalence relation on $\overline{G}$:

$$a \sim_G b \Leftrightarrow \forall t > 0 \text{ in } M, \exists a_t, b_t \in G, \text{ such that } \\ a_t \in \mathcal{B}_a(t),\ b_t \in \mathcal{B}_b(t) \text{ and } a_t \ominus b_t \in \mathcal{B}_0(t).$$

Clearly, $\forall a, b \in G$, $a \sim_G b \Leftrightarrow a = b$.
We can assume that $G \subseteq \overline{V}$:

Lemma 4.15. *G is definably isomorphic to a group G' with $G' \subseteq \overline{V^{G'}}$.*

Proof. Since V is large in G, it is everywhere dense, so $G \subseteq \overline{V}^t$. This implies that $\forall a \in G, \exists b \in \overline{V}$, such that $a \sim_G b$. Indeed, for any $a \in G$ and any $t > 0$ in M, there is $b_t \in V$, so that $a \ominus b_t \in \mathcal{B}_0(t)$. Since $V \subseteq G$ is bounded (Remark 3.7), $\overline{V}$ is closed and bounded. Thus $b := \lim_{t\to 0} b_t \in \overline{V}$, by [PeS]. We have $a \sim_G b$. Now, by definable choice, there is a definable subset Y of $\overline{V}$ of representatives for $\sim_G$ $\big($by considering the restriction of $\sim_G$ on $\overline{V} \times \overline{V}\big)$. Since each class can contain only one element of G, the definable function:

$$f : G \ni x \mapsto \text{the unique element } a \text{ with } x \sim_G a \in Y \subseteq \overline{V},$$

is a definable bijection between G and Y. We can let G' be the topological group with domain Y and structure the one induced by f, according to Remark 2.2. □

Note that now $\mathrm{bd}(V) = \mathrm{bd}(G)$. Indeed, since $V \subseteq G \subseteq \overline{V}$, we have $\overline{V} \subseteq \overline{G} \subseteq \overline{V}$ and $\mathrm{Int}(V) \subseteq \mathrm{Int}(G) \subseteq \mathrm{Int}(\overline{V}) = \mathrm{Int}(V)$, that is, $\overline{G} = \overline{V}$ and $\mathrm{Int}(G) = \mathrm{Int}(V)$.

Definition 4.16. Let $\gamma : [p, q] \subseteq M \to G$ be a t-path. An element $w \in M^n$, $w \neq 0$, is said to be a *jump (vector) of* γ if there is some $t_0 \in [p, q]$ such that

$$w = \gamma(t_0) - \lim_{t\to t_0^-} \gamma(t) \quad \text{or} \quad w = \lim_{t\to t_0^+} \gamma(t) - \gamma(t_0). \tag{4}$$

We say that γ *jumps at* t_0.

An element $w \in M^n$ is called a *jump vector (for G)* if it is the jump of some t-path.

Remark 4.17. (i) One can see that: w is a jump of some t-path $\Leftrightarrow \exists$ distinct $a, b \in \mathrm{bd}(V)$, such that $a \sim_G b$ and $w = b - a$. Thus, *the set of all jump vectors is a definable subset of* M^n.

(ii) Since γ is a t-path, $\lim^t_{t\to t_0^-} \gamma(t) = \gamma(t_0) = \lim^t_{t\to t_0^+} \gamma(t)$, contrasting (4). The last equation is equivalent to $\lim_{z\to 0} \big[\gamma(t_0 - z) \ominus \gamma(t_0 + z)\big] = 0$.

(iii) In case $\gamma : [0, p] \to G$ is a t-path with no jumps, then it is a path in M^n as well and it has the form $u + \varepsilon(t)$, where $u = \gamma(0)$, and $\varepsilon(t) = \gamma(t) - u$ is a path in M^n with $\varepsilon(0) = 0$. Conversely, if a t-path has the form $u + \varepsilon(t)$ for some path $\varepsilon(t)$ in M^n, then it has no jumps. For example, every t-path in V is of this form, as the $\mathcal{M}$- and t- topologies coincide on V.

Lemma 4.18. *Let $u, v \in V$ such that $u \oplus v \in V$, and $u + \varepsilon(t) : [0, p] \to V$, $\varepsilon(0) = 0$, a t-path. Then, $\exists t_0 \in (0, p]$, such that $\forall t \in [0, t_0]$,*

$$\big(u + \varepsilon(t)\big) \oplus v = (u \oplus v) + \varepsilon(t).$$

Proof. Let $r > 0$ be as in Corollary 4.13 and choose $t_0 \in (0, p]$ such that $\forall t \in [0, t_0]$, $u + \varepsilon(t) \in \mathcal{B}_u(r)$. □

Lemma 4.19. *Let $\gamma(t) = u + \varepsilon(t) : [0,p] \to V$, $\varepsilon(0) = 0$, be a t-path, such that $\forall t \in [0,p]$, $\varepsilon(t) \in V$. Then:*

$$\big(u + \varepsilon(p)\big) \ominus u = \varepsilon(p).$$

Proof. Consider the function $f : G \ni x \mapsto x - (x \ominus u) \in M^n$. By Lemma 4.18, f is locally constant on $\mathrm{Im}(\gamma)$. Indeed, first observe that $\forall s \in [0,p]$, $\exists z > 0$, such that $\forall t \in [s-z, s+z] \cap [0,p]$,

$$\big(u + \varepsilon(t)\big) \ominus u = \big(u + \varepsilon(s) + \varepsilon(t) - \varepsilon(s)\big) \ominus u = \big[\big(u + \varepsilon(s)\big) \ominus u\big] + \varepsilon(t) - \varepsilon(s).$$

Then, $\forall t \in [s-z, s+z]$, $f\big(u+\varepsilon(t)\big) = u + \varepsilon(t) - \big[\big(u+\varepsilon(t)\big) \ominus u\big] = u + \varepsilon(s) - \big[\big(u+\varepsilon(s)\big) \ominus u\big] = f\big(u+\varepsilon(s)\big)$.

It follows that f is constant on $\mathrm{Im}(\gamma)$ and equal to $u - (u \ominus u) = u$. Thus, $\forall t \in [0,p]$, $u + \varepsilon(t) - \big[\big(u+\varepsilon(t)\big) \ominus u\big] = u$, that is, $\big(u+\varepsilon(t)\big) \ominus u = \varepsilon(t)$. □

Lemma 4.20. *Let $u + \varepsilon(t) : [0,p] \to G$, $\varepsilon(0) = 0$, be a t-path that does not jump at $t = 0$, such that $\forall t \in (0,p]$, $u + \varepsilon(t) \in V$, and $\forall s,t \in [0,p]$, $\varepsilon(s) - \varepsilon(t) \in V$. Then:*

$$\big(u + \varepsilon(p)\big) \ominus u = \varepsilon(p).$$

Proof. By Lemma 4.19, we have $\forall t \in (0,p]$, $\big(u + \varepsilon(t) + \varepsilon(p) - \varepsilon(t)\big) \ominus \big(u + \varepsilon(t)\big) = \varepsilon(p) - \varepsilon(t)$, that is,

$$\big(u + \varepsilon(p)\big) \ominus \big(u + \varepsilon(t)\big) = \varepsilon(p) - \varepsilon(t).$$

On the other hand, since for all (small) $t \in [0,p]$, $\varepsilon(p) - \varepsilon(t) \in V$, the limits of the above expression with respect to the t- and $\mathcal{M}$- topologies as $t \to 0$ must coincide and be equal to $\varepsilon(p)$:

$$\lim_{t\to 0}{}^{t}\big[\big(u + \varepsilon(p)\big) \ominus \big(u + \varepsilon(t)\big)\big] = \lim_{t\to 0}\big(\varepsilon(p) - \varepsilon(t)\big) = \varepsilon(p).$$

Since $u+\varepsilon(t) : [0,p] \to G$ does not jump at $t = 0$, we also have $\lim^t_{t\to 0}\big(u+\varepsilon(t)\big) = u$. It follows, $\big(u + \varepsilon(p)\big) \ominus u = \lim^t_{t\to 0}\big[\big(u + \varepsilon(p)\big) \ominus \big(u + \varepsilon(t)\big)\big] = \varepsilon(p)$. □

Lemma 4.21. *Let $u + \varepsilon(t) : [0,p] \to G$, $\varepsilon(0) = 0$, be a t-path that does not jump at $t = 0$. Then: $\exists t_0 \in (0,p]$, such that $\forall t \in [0,t_0]$,*

$$\big(u + \varepsilon(t)\big) \ominus u = \varepsilon(t).$$

Proof. By curve selection, since $G \subseteq \overline{V}$ and $u + \varepsilon(t)$ does not jump at $t = 0$, it is not hard to see that there is some $t_0 \in (0,p]$ and, for all $s \in [0,t_0]$, a t-path $u + \delta_s(t) : [0,s] \to G$ with no jumps such that:

(i) $\delta_s(0) = 0$, $\delta_s(s) = \varepsilon(s)$, and $\forall t \in (0,s)$, $u + \delta_s(t) \in V$, and

(ii) $\forall t_1, t_2 \in [0,s]$, $\delta_s(t_1) - \delta_s(t_2) \in V$.

Now, by Lemma 4.20, $\forall s \in [0,t_0]$, $\forall t \in [0,s)$,

$$\big(u + \delta_s(t)\big) \ominus u = \delta_s(t) \in V_0.$$

Since $u + \delta_s(t)$ does not jump at $t = s$,

$$\big(u + \delta_s(s)\big) \ominus u = \lim_{t\to s}{}^{t}\big[\big(u + \delta_s(t)\big) \ominus u\big] = \lim_{t\to s}{}^{t}\delta_s(t) = \delta_s(s).$$

We have shown: $\forall s \in [0,t_0]$, $\varepsilon(s) = \delta_s(s) = \big(u + \delta_s(s)\big) \ominus u = \big(u + \varepsilon(s)\big) \ominus u$. □

Lemma 4.22. *Let $u, v \in G$ and $u + \varepsilon(t) : [0,p] \to G$, $\varepsilon(0) = 0$, a t-path that does not jump at $t = 0$, such that*

(i) $(u \oplus v) + \varepsilon(t)$ is a t-path,

or

(ii) $\big(u + \varepsilon(t)\big) \oplus v$ is a t-path that does not jump at $t = 0$.

Then: $\exists t_0 \in (0,p]$, such that $\forall t \in [0, t_0]$,

$$\big(u + \varepsilon(t)\big) \oplus v = (u \oplus v) + \varepsilon(t).$$

Proof. (i) Notice, by Remark 4.17(iii), $(u \oplus v) + \varepsilon(t)$ does not jump at $t = 0$. Applying Lemma 4.21 both to $u + \varepsilon(t)$ and to $(u \oplus v) + \varepsilon(t)$, we obtain: $\exists t_0 \in (0,p]\, \forall t \in [0, t_0]$,

$$\big(u + \varepsilon(t)\big) \ominus u = \varepsilon(t) = [(u \oplus v) + \varepsilon(t)] \ominus (u \oplus v).$$

(ii). Since $\big(u + \varepsilon(t)\big) \oplus v$ does not jump at $t = 0$, there exists some $s \in (0,p]$, such that $\forall t \in [0, s]$, $\big(u + \varepsilon(t)\big) \oplus v = (u \oplus v) + d_\varepsilon(t)$ for some path $d_\varepsilon(t)$ in M^n, that is, $[(u \oplus v) + d_\varepsilon(t)] \ominus (u \oplus v) = \big(u + \varepsilon(t)\big) \ominus u$. On the other hand, by Lemma 4.21, there is $t_0 \in (0, s]$, such that $\forall t \in [0, t_0]$,

$$[(u \oplus v) + d_\varepsilon(t)] \ominus (u \oplus v) = d_\varepsilon(t) \text{ and } \big(u + \varepsilon(t)\big) \ominus u = \varepsilon(t).$$

It follows that $\forall t \in [0, t_0]$, $d_\varepsilon(t) = \varepsilon(t)$. □

Lemma 4.23. *Let $u, v \in G$, and $\gamma(t) = u + \varepsilon(t) : [0,p] \to G$, $\varepsilon(0) = 0$, be a t-path with no jumps, such that*

(i) $(u \oplus v) + \varepsilon(t)$ is a t-path,

or

(ii) $\big(u + \varepsilon(t)\big) \oplus v$ is a t-path with no jumps.

Then:

$$\big(u + \varepsilon(p)\big) \oplus v = (u \oplus v) + \varepsilon(p).$$

Proof. Notice, by Remark 4.17(iii), $(u \oplus v) + \varepsilon(t)$ has no jumps. Consider the function $f : G \ni x \mapsto x + v - (x \oplus v) \in G$. By Lemma 4.22, it follows that f is locally constant on $\mathrm{Im}(\gamma)$. Thus it is constant on $\mathrm{Im}(\gamma)$ and equal to $u + v - (u \oplus v)$. Hence for all $t \in [0,p]$, $u + \varepsilon(t) + v - \big[\big(u + \varepsilon(t)\big) \oplus v\big] = u + v - (u \oplus v)$, that is, $\big(u + \varepsilon(t)\big) \oplus v = (u \oplus v) + \varepsilon(t)$. □

By o-minimality, a t-path γ jumps at finitely many points $t_1, \dots, t_r$ of its domain at most. If $w_1, \dots, w_r$ are its jumps, their sum is denoted by

$$J_\gamma := \sum_{i=1}^{r} w_i.$$

Proposition 4.24. *Let $u, v \in G$, and $\gamma(t) = u + \varepsilon(t) : [0,p] \to G$, $\varepsilon(0) = 0$, be a t-path with no jumps. Then:*

$$\big(u + \varepsilon(p)\big) \oplus v = (u \oplus v) + \varepsilon(p) + J_{\gamma \oplus v}.$$

Proof. Assume that $\gamma(t) \oplus v$ has a jump w_i at t_i, for $1 \le i \le r$ and $0 = t_0 \le t_1 \le \dots \le t_r \le t_{r+1} = 1$. Let $w_0, w_{r+1} := 0$, and for all $i \in \{0, \dots, r+1\}$, $J_i := \sum_{k=0}^{i} w_k$, and $\gamma^i := \gamma \restriction_{[0,t_i]}$. By induction on i, we show that for all $i \in \{0, \dots, r+1\}$ the proposition is true for γ^i, that is,

$$\gamma(t_i) \oplus v = (u \oplus v) + \varepsilon(t_i) + J_{\gamma^i \oplus v}. \tag{5}$$

(5) is clearly true for $i = 0$. Now, assume that (5) holds for γ^i, for some $i \in \{0, \ldots, r\}$. We show that (5) holds for γ^{i+1}. If $t_i = t_{i+1}$ there is nothing to show, so assume $t_i < t_{i+1}$.

Claim. For all $s \in (t_i, t_{i+1})$,

$$\gamma(s) \oplus v = (u \oplus v) + \varepsilon(s) + J_i. \tag{6}$$

Proof of Claim. We first show

$$\lim_{t \to t_i^+} \big(\gamma(t) \oplus v\big) = (u \oplus v) + \varepsilon(t_i) + J_i. \tag{7}$$

Case 1: $w_i = \big(\gamma(t_i) \oplus v\big) - \lim_{t \to t_i^-} \big(\gamma(t) \oplus v\big)$. Then $\gamma(t_i) \oplus v = \lim_{t \to t_i^+} \big(\gamma(t) \oplus v\big)$, and $J_{\gamma^i \oplus v} = J_i$. By Inductive Hypothesis, (7) follows.

Case 2: $w_i = \lim_{t \to t_i^+} \big(\gamma(t) \oplus u\big) - \big(\gamma(t_i) \oplus v\big)$. Then, $J_{\gamma^i \oplus v} + w_i = J_i$, and by Inductive Hypothesis, (7) follows.

Now, for any t with $t_i < t < s$, Lemma 4.23(ii) gives $\big(u + \varepsilon(s)\big) \oplus v = \big(u + \varepsilon(t) + \varepsilon(s) - \varepsilon(t)\big) \oplus v = \big[\big(u + \varepsilon(t)\big) \oplus v\big] + \varepsilon(s) - \varepsilon(t)$. Therefore, $\big(u + \varepsilon(s)\big) \oplus v = \lim_{t \to t_i^+} \big[\big(u + \varepsilon(s)\big) \oplus v\big] = \lim_{t \to t_i^+} \big[\big(u + \varepsilon(t)\big) \oplus v\big] + \varepsilon(s) - \varepsilon(t_i)$. By (7), we have $\big(u + \varepsilon(s)\big) \oplus v = (u \oplus v) + \varepsilon(s) + J_i$, that is, (6) holds. This proves the Claim. □

We now show that (5) is true for γ^{i+1}. Taking limits from the left of t_{i+1} in equation (6) we get:

$$\lim_{s \to t_{i+1}^-} \big(\gamma(s) \oplus v\big) = (u \oplus v) + \varepsilon(t_{i+1}) + J_i. \tag{8}$$

Case 1: $w_{i+1} = \lim_{t \to t_{i+1}^+} \big(\gamma(t) \oplus v\big) - \big(\gamma(t_{i+1}) \oplus v\big)$. Then $\gamma(t_{i+1}) \oplus v =$
$= \lim_{t \to t_{i+1}^-} \big(\gamma(t) \oplus v\big)$ and $J_{\gamma^{i+1} \oplus v} = J_i$. By (8), equation (5) is true for γ^{i+1}.

Case 2: $w_{i+1} = \big(\gamma(t_{i+1}) \oplus v\big) - \lim_{t \to t_{i+1}^-} \big(\gamma(t) \oplus v\big)$. Then $J_{\gamma^{i+1} \oplus v} = J_i + w_{i+1}$, and by (8), again, (5) is true for γ^{i+1}. □

STEP II. A generic open n-parallelogram of G. Since V is large in G, it is also generic, by Fact 3.9(i). By Linear CDT, V is a finite union of linear cells, and by Lemma 3.10, one of them, call it Y, must be generic. By Fact 3.9(ii), Y has dimension n, and by Lemma 3.7, it is bounded. Therefore, by Lemma 3.6, $\overline{Y}$ is a finite union of closed n-parallelograms, say $W_1, \ldots, W_l$. For $i \in \{1, \ldots, l\}$, let $Y_i := Y \cap W_i$. Then $Y = Y_1 \cup \ldots \cup Y_l$. By Lemma 3.10 again, one of the Y_i's must be generic, say Y_1. Let $H := \mathrm{Int}(Y_1)$. Since on V the $\mathcal{M}$- and t- topologies coincide, $H = \mathrm{Int}(Y_1)^t$. By Fact 3.9(iii), H is generic. Since W_1 is a closed n-parallelogram and $\mathrm{Int}(W_1) = \mathrm{Int}(W_1 \cap \overline{Y}) = \mathrm{Int}(W_1 \cap Y) = \mathrm{Int}(Y_1) = H$, we have that H is an open n-parallelogram.

Let c be the center of H. By translation in M^n, we can assume that $c = 0$. Indeed, in Lemma 4.9 we could have let $f : G \ni x \mapsto (x \oplus c) - c \in M^n$. Since H is generic, $H \ominus c$ is generic, and thus $f(H \ominus c) = H - c$ is a generic open n-parallelogram of $f(G)$ centered at 0. To see that the $\mathcal{M}$- and t- topologies coincide on $H - c \subseteq f(G)$, consider the definable automorphism

$$\overline{f} : M^n \ni x \mapsto x - c \in M^n,$$

and notice moreover that $\overline{f} \restriction_G : G \to f(G)$ is in fact a homeomorphism, since for all $x \in G$, $\overline{f}(x) = f(x \ominus c)$.

Summarizing, we can assume that:

- *H is a generic, t-open, open n-parallelogram, with center 0, on which the $\mathcal{M}$- and t- topologies coincide.*

Since H is generic, it must have dimension n and, thus, the form:

$$H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\},$$

for some M-independent $\lambda_i \in D^n$ and positive $e_i \in M$.

Lemma 4.25. *Let $a, b \in H$, such that $a + b \in H$. Then there is a path $\varepsilon(t)$ in H from 0 to a, such that the path $\varepsilon(t) + b$ lies entirely in H, as well.*

Proof. We show it for *any* open m-parallelogram $H = \{\lambda_1 t_1 + \ldots + \lambda_m t_m : -e_i < t_i < e_i\} \subset M^n$, $0 < m \leq n$, for M-independent $\lambda_i \in D^n$ and positive $e_i \in M$, by induction on m.

$\mathbf{m = 1}$. Let $H = \{\lambda_1 t_1 : -e_1 < t_1 < e_1\}$ containing a, b and $a + b$. Assume $a = \lambda_1 t_{a1}$, for some $t_{a1} \in (-e_1, e_1)$. It is then easy to see that the path $[0, t_{a1}] \ni t \mapsto \varepsilon(t) := \lambda_1 t \in H$ satisfies the conclusion, by convexity of H and Lemma 3.4.

$\mathbf{m > 1}$. Let $H = \{\lambda_1 t_1 + \ldots + \lambda_m t_m : -e_i < t_i < e_i\}$ containing a, b and $a + b$, and let $a = \lambda_1 t_{a1} + \ldots + \lambda_m t_{am}$, $b = \lambda_1 t_{b1} + \ldots + \lambda_m t_{bm}$, for some $t_{ai}, t_{bi} \in (-e_i, e_i)$. Consider the open $(m-1)$-parallelogram $H' = \{\lambda_2 t_2 + \ldots + \lambda_m t_m : -e_i < t_i < e_i\}$, and let $a' := \lambda_2 t_{a2} + \ldots + \lambda_m t_{am}$, $b' := \lambda_2 t_{b2} + \ldots + \lambda_m t_{bm}$. By Inductive Hypothesis, there is a path ε' in H' from 0 to a', such that $b' + \varepsilon'(t)$ is a path in H' from b' to $a' + b'$. Let $\varepsilon(t)$ be the concatenation of ε' with the linear path $a' + \lambda_1 t$, $t \in [0, t_{a1}]$, from a' to a. It is then easy to check, using the convexity of H and Lemma 3.4, that both $\varepsilon(t)$ and $b + \varepsilon(t)$ lie entirely in H. □

Since the two topologies coincide on H, the paths $\varepsilon(t)$ and $b + \varepsilon(t)$ from Lemma 4.25 are also t-paths.

Lemma 4.26. *Let $x_1, \ldots, x_l \in H$ be such that for any subset σ of $\{1, \ldots, l\}$, $\sum_{j \in \sigma} x_j \in H$. Then, $x_1 + \ldots + x_l = x_1 \oplus \ldots \oplus x_l$.*

Proof. By induction on l.

$\mathbf{l = 2}$. Let $a = x_1$, $b = x_2$, and $\gamma(t) = \varepsilon(t)$ as in Lemma 4.25. Then, by Lemma 4.23(i), for $u = 0$ and $v = b = x_2$, we have: $x_1 \oplus x_2 = (0 \oplus x_2) + x_1 = x_1 + x_2$.

$\mathbf{l > 2}$. $x_1 + \ldots + x_l = x_1 + (x_2 + \ldots + x_l) = x_1 \oplus (x_2 \oplus \ldots \oplus x_l) = x_1 \oplus \ldots \oplus x_l$. □

Lemma 4.27. *For every $x_1, \ldots, x_l, y_1, \ldots, y_m \in H$, if $x_1 + \ldots + x_l = y_1 + \ldots + y_m$, then $x_1 \oplus \ldots \oplus x_l = y_1 \oplus \ldots \oplus y_m$.*

Proof. Assume $x_1 + \ldots + x_l = y_1 + \ldots + y_m$, $x_i, y_i \in H$. We want to show $x_1 \oplus \ldots \oplus x_l = y_1 \oplus \ldots \oplus y_m$. Clearly, by convexity of H, for any subset σ of $\{1, \ldots, l\}$, $\sum_{i \in \sigma} \frac{x_i}{l} \in H$, and therefore $\sum_{i \in \sigma} \frac{x_i}{lm} \in H$. Similarly, for any subset τ of $\{1, \ldots, m\}$, $\sum_{j \in \tau} \frac{y_j}{m} \in H$ and $\sum_{j \in \tau} \frac{y_j}{lm} \in H$. By Lemma 4.26, on the one hand we have $\frac{x_1}{lm} \oplus \ldots \oplus \frac{x_l}{lm} = \frac{x_1}{lm} + \ldots + \frac{x_l}{lm} = \frac{y_1}{lm} + \ldots + \frac{y_m}{lm} = \frac{y_1}{lm} \oplus \ldots \oplus \frac{y_m}{lm}$, and, on the other, $\underbrace{\frac{x_i}{lm} \oplus \ldots \oplus \frac{x_i}{lm}}_{lm\text{-times}} = x_i$ and $\underbrace{\frac{y_j}{lm} \oplus \ldots \oplus \frac{y_j}{lm}}_{lm\text{-times}} = y_j$, for every i, j. Thus, $x_1 \oplus \ldots \oplus x_l =$

$$\bigoplus_{1\leq i\leq l}\Big(\underbrace{\frac{x_i}{lm}\oplus\ldots\oplus\frac{x_i}{lm}}_{lm\text{-times}}\Big) = \bigoplus_{lm\text{-times}}\left(\tfrac{x_1}{lm}\oplus\ldots\oplus\tfrac{x_l}{lm}\right) = \bigoplus_{lm\text{-times}}\left(\tfrac{y_1}{lm}\oplus\ldots\oplus\tfrac{y_m}{lm}\right) = \bigoplus_{1\leq i\leq m}\Big(\underbrace{\frac{y_i}{lm}\oplus\ldots\oplus\frac{y_i}{lm}}_{lm\text{-times}}\Big) = y_1\oplus\ldots\oplus y_m.$$ □

Lemma 4.28. *Let $H_1 := \frac{1}{2}H = \{\frac{1}{2}x : x\in H\}$. Then H_1 is generic.*

Proof. We show that finitely many $\oplus$-translates of H_1 cover H. By Lemma 4.26, it suffices to find finitely many $a_i\in H$, such that $H = \bigcup_i(a_i+H_1)$. Let $H = \{\lambda_1t_1+\ldots+\lambda_nt_n : -e_i<t_i<e_i\}$. Then $H_1 = \{\lambda_1t_1+\ldots+\lambda_nt_n : -\frac{e_i}{2}<t_i<\frac{e_i}{2}\}$. It is a routine to check that $H = \bigcup_{i=1}^{2^n}(a_i+H_1)$, where the a_i's are the corners of H_1. □

Lemma 4.29. *There is $K\in\mathbb{N}$, such that $G = \underbrace{H\oplus\ldots\oplus H}_{K-times}$.*

Proof. Let H_1 as in Lemma 4.28. Assume that for $K\in\mathbb{N}$, $\{a_i\oplus H_1\}_{\{1\leq i\leq K\}}$ covers G. Since G is t-connected (and H_1 is t-open), for any $x\in G$, one can find $0 = x_0, x_1,\ldots,x_l = x\in G$, $l\leq K$, such that $\forall i\in\{1,\ldots,l-1\}$, after perhaps reordering $\{a_i\oplus H_1\}_{\{1\leq i\leq K\}}$, $x_i\in(a_i\oplus H_1)\cap(a_{i+1}\oplus H_1)$, $0\in a_1\oplus H_1$, and $x\in a_l\oplus H_1$. Then, for $h_i := x_i\ominus x_{i-1}\in H$, $1\leq i\leq l$, we have: $x = h_1\oplus\ldots\oplus h_l$. □

Definition 4.30. Let U be the subgroup of M^n generated by H, that is,

$$U := \bigcup_{k<\omega} H^k,$$

where $H^k := \underbrace{H+\ldots+H}_{k-\text{times}}$. By Lemma 4.27, the following function $\phi : U\to G$ is well-defined. For all $x\in U$, if $x = x_1+\ldots+x_k$, $x_i\in H$, then

$$\phi(x) = x_1\oplus\ldots\oplus x_k.$$

$\mathcal{U} = \langle U, +_{\restriction U}, 0\rangle$ is a $\bigvee$-definable group. Easily, convexity of H implies convexity of U. Moreover:

Proposition 4.31. *ϕ is a t-continuous group homomorphism from $\mathcal{U}$ onto G.*

Proof. ϕ is a group homomorphism, because if $x = x_1+\ldots+x_l$ and $y = y_1+\ldots+y_m$, with $x_i, y_i\in H$, then $\phi(x+y) = \phi(x_1+\ldots+x_l+y_1+\ldots+y_m) = x_1\oplus\ldots\oplus x_l\oplus y_1\oplus\ldots\oplus y_m = \phi(x)\oplus\phi(y)$. It is onto, by Lemma 4.29. Since $\oplus$ is t-continuous, so is ϕ. □

Thus, if we let $L := \ker(\phi)$, we know that $\mathcal{U}/L\cong G$ as abstract groups.

STEP III. L is a lattice of rank n. We show that L is a lattice generated by n $\mathbb{Z}$-independent elements of M^n, namely, by some $\mathbb{Z}$-linear combinations of jump vectors for G. Recall that (Remark 4.17(i)) $w\in M^n$ is a jump vector if and only if there are distinct $a, b\in \mathrm{bd}(V)$ such that $a\sim_G b$ and $w = b-a$. The following is a consequence of the local analysis from Step I.

Lemma 4.32. *There are only finitely many jump vectors.*

Proof. Since the set of all jump vectors is definable, if there were infinitely many jump vectors, by o-minimality, one of the following should be true:

(A) there exists a non-constant path γ on $\mathrm{bd}(V)$, such that all points in $\mathrm{Im}(\gamma)$ are $\sim_G$-equivalent,

(B) there exist two disjoint non-constant paths γ and δ on $\mathrm{bd}(V)$, such that every element a in $\mathrm{Im}(\gamma)$ is $\sim_G$-equivalent with a unique element b_a in $\mathrm{Im}(\delta)$, and vice versa, and all jump vectors $w_a = b_a - a$, $a \in \mathrm{Im}(\gamma)$, are distinct.

Assume (A) holds. By o-minimality again, we can assume that $\gamma(t) = a + \varepsilon(t) : [0, p] \to M^n$, for some path $\varepsilon(t)$ in H with $\varepsilon(0) = 0$ and $\varepsilon := \varepsilon(p) \neq 0$. Moreover, we can assume that there is a path $\rho(s) : [0, q] \to M^n$, with $\rho(0) = 0$, such that $\forall s > 0$, $a + \rho(s)$ and $a + \varepsilon + \rho(s)$ are in G, and $a + \rho(s) + \varepsilon(t) : [0, p] \to G$ is a t-path, with no jumps, from $a + \rho(s)$ to $a + \rho(s) + \varepsilon$. By Lemma 4.23(i), we have that for all $s \in (0, p]$,

$$\big(a + \rho(s) + \varepsilon\big) \ominus \big(a + \rho(s)\big) = \varepsilon.$$

Thus $\lim_{s\to 0} \big[\big(a + \rho(s) + \varepsilon\big) \ominus \big(a + \rho(s)\big)\big] = \varepsilon \neq 0$, contradicting the fact that $a \sim_G a + \varepsilon$.

Now assume (B) holds and, without loss of generality, let $\gamma(t) = a + \varepsilon(t) : [0, p_\gamma] \to M^n$, for some path $\varepsilon(t)$ in H with $\varepsilon(0) = 0$ and $\varepsilon := \varepsilon(p_\gamma) \neq 0$. Let also $\delta(t) = b + \zeta(t) : [0, p_\delta] \to M^n$, for $b \sim_G a$ and some path $\zeta(t)$ in H with $\zeta(0) = 0$ and $\zeta := \zeta(p_\delta) \neq 0$. As before, we can assume that there is a path $\rho(s) : [0, q] \to M^n$, with $\rho(0) = 0$, such that $\forall s > 0$, $a+\rho(s)$ and $a+\varepsilon+\rho(s)$ are in G, and $a+\rho(s)+\varepsilon(t) : [0, p_\gamma] \to G$ is a t-path, with no jumps, from $a+\rho(s)$ to $a+\rho(s)+\varepsilon$. Similarly, we can assume that there is a path $\sigma(s) : [0, q] \to M^n$, with $\sigma(s) = 0$, such that $\forall s > 0$, $b+\sigma(s)$ and $b+\zeta+\sigma(s)$ are in G, and $b+\sigma(s)+\zeta(t) : [0, p_\delta] \to G$ is a t-path, with no jumps, from $b+\sigma(s)$ to $b+\sigma(s)+\zeta$. We show that if $a+\varepsilon \sim_G b+\zeta$, then $\varepsilon = \zeta$, which contradicts the fact that all jump vectors from $\mathrm{Im}(\gamma)$ to $\mathrm{Im}(\delta)$ are distinct. As before, we have that for any $s \in (0, p_\gamma] \cap (0, p_\delta]$,

$$\big(a + \rho(s) + \varepsilon\big) \ominus \big(a + \rho(s)\big) = \varepsilon \text{ and } \big(b + \sigma(s) + \zeta\big) \ominus \big(b + \sigma(s)\big) = \zeta.$$

On the other hand, since $a \sim_G b$ and $a + \varepsilon \sim_G b + \zeta$,

$$\lim_{s\to 0} \big[\big(a + \rho(s)\big) \ominus \big(b + \sigma(s)\big)\big] = 0 \text{ and } \lim_{s\to 0} \big[\big(a + \varepsilon + \rho(s)\big) \ominus \big(b + \zeta + \sigma(s)\big)\big] = 0.$$

Since in a t-neighborhood of 0 the $\mathcal{M}$- and t- topologies coincide,

$$\lim_{s\to 0}{}^{t} \big[\big(a + \rho(s)\big) \ominus \big(b + \sigma(s)\big)\big] = 0 \text{ and } \lim_{s\to 0}{}^{t} \big[\big(a + \varepsilon + \rho(s)\big) \ominus \big(b + \zeta + \sigma(s)\big)\big] = 0,$$

and, thus,

$$\begin{aligned}
\varepsilon \ominus \zeta &= \lim_{s\to 0}{}^{t} (\varepsilon \ominus \zeta) \\
&= \lim_{s\to 0}{}^{t} \big[\big(a + \varepsilon + \rho(s)\big) \ominus \big(a + \rho(s)\big) \ominus \big(b + \zeta + \sigma(s)\big) \oplus \big(b + \sigma(s)\big)\big] \\
&= \lim_{s\to 0}{}^{t} \big[\big(a + \varepsilon + \rho(s)\big) \ominus \big(b + \zeta + \sigma(s)\big)\big] \ominus \lim_{s\to 0}{}^{t} \big[\big(a + \rho(s)\big) \ominus \big(b + \sigma(s)\big)\big] \\
&= 0,
\end{aligned}$$

hence $\varepsilon = \zeta$. □

Let $\{w_1, \ldots, w_l\}$ be the set of all jump vectors for G.

Lemma 4.33. $\ker(\phi) \subseteq \mathbb{Z}w_1 + \ldots + \mathbb{Z}w_l$.

Proof. Let $x = x_1 + \ldots + x_m \in \ker(\phi) \subseteq U$, with $x_i \in H$. For all $i \in \{1, \ldots, m\}$, let $x_i(t)$ be a path in H from 0 to x_i. By Proposition 4.24,

$$\phi(x) = x_1 \oplus \ldots \oplus x_m = x_1 + \ldots + x_m + J_\gamma,$$

where γ is the t-loop $\big(x_1(t)\big) \vee \big(x_1 \oplus x_2(t)\big) \vee \ldots \vee \big(x_1 \oplus \ldots \oplus x_{m-1} \oplus x_m(t)\big)$ from 0 to $x_1 \oplus \ldots \oplus x_m = \phi(x_1 + \ldots + x_m) = \phi(x) = 0$. We have: $x = -J_\gamma \in \mathbb{Z}w_1 + \ldots + \mathbb{Z}w_l$. □

A subgroup of the torsion-free group M^n is torsion-free. Thus, $\mathbb{Z}w_1 + \ldots + \mathbb{Z}w_l$ is a finitely generated torsion-free abelian subgroup of M^n, and therefore it is free. Since $\ker(\phi) \leqslant \mathbb{Z}w_1 + \ldots + \mathbb{Z}w_l$, it follows that $\ker(\phi)$ is a free abelian subgroup of U generated by k $\mathbb{Z}$-independent elements, for some $k \leq l$. (The reader is referred to [Lang, Chapter I] for any of the above assertions.) In Claims 4.36 and 4.37 below we show that $k = n$.

Before that, we use H to obtain a 'standard part' map from U to $\mathbb{R}^n$. Recall, H is a generic open n-parallelogram of the form:

$$H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\},$$

for some M-independent $\lambda_i \in D^n$, and positive $e_i \in M$. The following map must then be one-to-one:

$$\Theta : M^n \ni (t_1, \ldots, t_n) \mapsto \lambda_1 t_1 + \ldots + \lambda_n t_n \in M^n.$$

Let $u = h_1 + \ldots + h_m \in U$, with $h_j \in H$. For $j \in \{1, \ldots, m\}$, it must be $h_j = \lambda_1 t_1^j + \ldots + \lambda_n t_n^j \in H$, for some $t_i^j \in (-e_i, e_i)$. Thus, $u = \lambda_1 \left(\sum_{j=1}^m t_1^j\right) + \ldots + \lambda_n \left(\sum_{j=1}^m t_n^j\right)$ and $\Theta\bigg(\left(\sum_{j=1}^m t_1^j, \ldots, \sum_{j=1}^m t_n^j\right)\bigg) = u$, with $-me_i < \sum_{j=1}^m t_i^j < me_i$. This shows that $U \subseteq \Theta(M^n)$ and, in particular, that for every $u \in U$, $\Theta^{-1}(u) = (u_1, \ldots, u_n) \in M^n$ with $\forall i \exists q \in \mathbb{Z}$, $-qe_i < u_i < qe_i$. We define the *standard part* map from U to $\mathbb{R}^n$, as follows. We let

$$st(u) := \big(st(u_1), \ldots, st(u_n)\big) \in \mathbb{R}^n,$$

where each $st_i(u_i)$ is defined by the Dedekind cut $\{q \in \mathbb{Q} : qe_i < u_i\}$, $\{q \in \mathbb{Q} : u_i \leq qe_i\}$, that is,

$$st(u_i) := \sup\{q \in \mathbb{Q} : qe_i < u_i\}.$$

Easily, st is a group homomorphism from $\langle U, +_{\restriction U}, 0\rangle$ onto $\langle \mathbb{R}^n, +, 0\rangle$, where, henceforth, $+$ denotes the usual real addition whenever it applies to real numbers.

We let

$$\forall x \in U,\ ||x|| := |st(x)|_{\mathbb{R}},$$

where $|\cdot|_{\mathbb{R}}$ is the Euclidean norm in $\mathbb{R}^n$. It is easy to check that $||\cdot||$ is a 'seminorm on U over $\mathbb{Q}$', that is:

(i) $\forall x, y \in U$, $||x + y|| \leq ||x|| + ||y||$, and (ii) $\forall q \in \mathbb{Q}, \forall x \in U$, $||qx|| = |q|\,||x||$.

Lemma 4.34. *For all $x \in U$ and $m \in \mathbb{N}$,*

$$x \in H^m \Leftrightarrow ||x||_H < m\sqrt{n}.$$

Proof. Let $\Theta^{-1}(x) = (x_1, \ldots, x_n) \in M^n$. Then, $x \in H^m \Leftrightarrow \forall i, -me_i < x_i < me_i \Leftrightarrow st(x) \in [-m, m]^n \subset \mathbb{R}^n \Leftrightarrow |st(x)|_{\mathbb{R}} < \sqrt{nm^2} = m\sqrt{n}$. □

Let us also collect two easy but helpful facts about $\ker(\phi)$:

Lemma 4.35. *(i)* $\ker(\phi) \cap H = \{0\}$.
(ii) Let K be as in Lemma 4.29. Then $\forall x \in U, \exists y \in H^K$, $y - x \in \ker(\phi)$.

Proof. (i) For all $x \in H$, $\phi(x) = x$.
(ii) For $x \in U$, since $\phi(x) \in G$, there are $x_1, \ldots, x_K \in H$, such that $\phi(x) = x_1 \oplus \ldots \oplus x_K$. Clearly, if $y = x_1 + \ldots + x_K \in H^K$, then $\phi(x) = \phi(y)$. □

We are now ready to compute the rank of $L = \ker(\phi)$. Fix a set $\{v_1, \ldots, v_k\}$ of generators for L.

Claim 4.36. *$k \geq n$.*

Proof. Assume, towards a contradiction, that $k < n$. For any $a \in U$, let $S_a := a + \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_k$. Let K be as in Lemma 4.29.

Subclaim. *There is $a \in U$, such that $S_a \cap H^K = \emptyset$.*

Proof of Subclaim. By Lemma 4.34, it suffices to show that there is $a \in U$, such that $\forall l_1, \ldots, l_k \in \mathbb{N}$, $||a + l_1 v_1 + \ldots + l_k v_k|| \geq K\sqrt{n}$. But,

$$||a + l_1 v_1 + \ldots + l_k v_k|| = |st(a) + l_1 st(v_1) + \ldots + l_k st(v_k)|_{\mathbb{R}},$$

and, since $k < n$, there is $\bar{a} \in \mathbb{R}^n$ such that $\forall l_1, \ldots, l_k \in \mathbb{N}$,

$$|\bar{a} + l_1 st(v_1) + \ldots + l_k st(v_k)|_{\mathbb{R}} \geq K\sqrt{n}.$$

(This is true for *any* number $K\sqrt{n}$.) We can take a to be any element in $st^{-1}(\bar{a})$. □

This contradicts Lemma 4.35(ii). □

Claim 4.37. *$k \leq n$.*

Proof. Notice that $st(L)$ is a lattice in $\mathbb{R}^n$ contained in $\mathbb{Z}st(v_1) + \ldots + \mathbb{Z}st(v_k)$.

Subclaim. *$st(L)$ has rank k.*

Proof of Subclaim. Clearly, $st(L)$ has rank at most k. If $st(L)$ has rank less than k, then for some $l_1, \ldots, l_k \in \mathbb{Z}$, not all zero, $l_1 st(v_1) + \ldots + l_k st(v_k) = 0$. Since $st : U \to \mathbb{R}^n$ is a group homomorphism, $st(l_1 v_1 + \ldots + l_k v_k) = 0$. Thus, $l_1 v_1 + \ldots + l_k v_k \in H$. On the other hand, $\phi(l_1 v_1 + \ldots + l_k v_k) = 0$. Hence, by Lemma 4.35(i), we have $l_1 v_1 + \ldots + l_k v_k = 0$, contradicting the fact that L has rank k. □

Lemma 4.35(i) also gives us that $st(L)$ is discrete: $st\left(\frac{1}{2}H\right)$ is an open neighborhood of 0 that contains no other elements from $st(L)$. But it is a classical fact that every discrete subgroup of $\mathbb{R}^n$ is generated by $\leq n$ elements (see [BD, Chapter I, Lemma 3.8], for example). Thus $k \leq n$. □

Proof of Theorem 1.4. For convenience, we collect main definitions and facts. In Step II, Definition 4.30, we defined the convex $\bigvee$-definable subgroup $U = \langle U, +_{\restriction U}, 0\rangle$ of M^n, generated by a generic, t-open, open n-parallelogram $H \subseteq G$ centered at 0. We also let $\phi : U \to G$ be such that $(\forall k \in \mathbb{N})(\forall x = x_1 + \ldots + x_k, h_i \in H)[\phi(x) = x_1 \oplus \ldots \oplus x_k]$. We showed that ϕ is an onto homomorphism, Proposition 4.31, and in Step III, that $L := \ker(\phi) \leqslant U$ is a lattice of rank n, Claims 4.36 and 4.37. We have $U/L \cong G$ as abstract groups. Notice, ϕ restricted to a definable subset of U is a definable map.

Let $\Sigma := H^K$, where K is as in Lemma 4.29. Clearly, Σ is definable, and thus $\phi_{\restriction \Sigma}$ is definable. Moreover, E^{Σ}_L is definable, since, for all $x, y \in \Sigma$, we have

$xE_L^\Sigma y \Leftrightarrow x - y \in L \Leftrightarrow \phi_{\restriction \Sigma}(x) = \phi_{\restriction \Sigma}(y)$. By Lemma 4.35(ii), Σ contains a complete set S of representatives for E_L^U, and, by definable choice, there is a definable such set S. By Claim 2.7(ii), $U/L = \langle S, +_S \rangle$ is a definable quotient group. The restriction of ϕ on S is a definable group isomorphism between $\langle S, +_S \rangle$ and G. By Remark 2.2(ii), we are done. □

The following is immediate.

Corollary 4.38. *For every $k \in \mathbb{N}$, the k-torsion subgroup of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$.*

5. On Pillay's conjecture

In this section we show Pillay's conjecture in the present context, that is, for $\mathcal{M}$ a saturated ordered vector space over an ordered division ring. The reader is referred to [Pi2] for any terminology.

Proposition 5.1 (Pillay's Conjecture). *Let G be an n-dimensional, definably compact and t-connected group, definable in $\mathcal{M}$. Then, there is a smallest type-definable subgroup G^{00} of G of bounded index such that G/G^{00}, when equipped with the logic topology, is a compact Lie group of dimension n.*

Proof. Recall that H is an open n-parallelogram with center 0. For $n \in \mathbb{N}$, we define H_n inductively as follows: $H_0 = H$, and $H_{n+1} = \frac{1}{2}H_n$. By Lemma 4.26, $B = \bigcap_{n<\omega} H_n$ is then a type-definable subgroup of G. As in the proof of Lemma 4.28, one can show that for all n, finitely many $\oplus$-translates of H_{n+1} cover H_n, and thus, inductively, finitely many $\oplus$-translates of H_{n+1} cover G. It follows that B has bounded index in G. Note also that B is torsion-free: if $m \in \mathbb{N}$ and $x \in B \setminus \{0\}$, then $x \in H_m$, and thus, by Lemma 4.26, $\underbrace{x \oplus \ldots \oplus x}_{m-\text{times}} = mx \neq 0$.

By [BOPP], there is a smallest type-definable subgroup G^{00} of bounded index, which is divisible, and G/G^{00} with the logic topology is a connected compact abelian Lie group. By [BOPP, Corollary 1.2], a torsion-free type-definable subgroup of G of bounded index is equal to G^{00}, hence $B = G^{00}$. Since G^{00} is torsion-free and divisible, it follows that for all k, the k-torsion subgroup of G/G^{00} is isomorphic to the k-torsion subgroup of G, which is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$, by Corollary 4.38. Thus, G/G^{00} is isomorphic to the real n-torus and has dimension n. □

6. O-minimal fundamental group

The o-minimal fundamental group is defined as in the classical case (see [Hat], for example) except that all paths and homotopies are definable. The following is a restatement of the definition given in [BO2], where $\mathcal{M}$ expanded an ordered field. A different definition of the o-minimal fundamental group was given in [Ed3] for a locally definable group in any o-minimal structure, using locally definable covering homomorphisms. In [EdEl], the two definitions are shown to be equivalent for a group definable in any o-minimal expansion of an ordered group.

The next two definitions run in parallel with respect to the product topology of M^n and the t-topology on G. Notice that until Lemma 6.8, $\mathcal{M}$ can be any o-minimal expansion of an ordered group and G any group definable in $\mathcal{M}$.

Definition 6.1 ([vdD], Chapter 8, (3.1))**.** Let $f, g : M^m \supseteq X \to M^n$ (G) be two definable (t-)continuous maps in M^n (in G). A *(t-)homotopy between f and g* is a definable (t-)continuous map $F(t,s) : X \times [0,q] \to M^n$ (G), for some $q > 0$ in M, such that $f = F_0$ and $g = F_q$, where $\forall s \in [0,q]$, $F_s := F(\cdot, s)$. We call f and g *(t-)homotopic*, denoted by $f \sim g$ ($f \sim_t g$).

Definition 6.2. Two (t-)paths $\gamma : [0,p] \to M^n$ (G), $\delta : [0,q] \to M^n$ (G), with $\gamma(0) = \delta(0)$ and $\gamma(p) = \delta(q)$, are called *(t-)homotopic* if there is some $t_0 \in [0, \min\{p,q\}]$, and a (t-)homotopy $F(t,s) : [0, \max\{p,q\}] \times [0,r] \to M^n$ (G), for some $r > 0$ in M, between

$$\gamma_{\restriction[0,t_0]} \vee \mathbf{c} \vee \gamma_{\restriction[t_0,p]} \text{ and } \delta \text{ (if } p \leq q\text{), or}$$

$$\delta_{\restriction[0,t_0]} \vee \mathbf{d} \vee \delta_{\restriction[t_0,q]} \text{ and } \gamma \text{ (if } q \leq p\text{),}$$

where $\mathbf{c}(t) = \gamma(t_0)$ and $\mathbf{d}(t) = \delta(t_0)$ are the constant paths with domain $[0, |p-q|]$.

If $\mathbb{L}(G)$ denotes the set of all t-loops that start and end at 0, then the restriction $\sim_t\restriction_{\mathbb{L}(G)\times\mathbb{L}(G)}$ is an equivalence relation on $\mathbb{L}(G)$. Let $\pi_1(G) := \mathbb{L}(G)/\sim_t$ and $[\gamma] :=$ the class of $\gamma \in \mathbb{L}(G)$.

It is clear that any two constant (t-)loops with image $\{0\}$ (but perhaps different domains) are (t-)homotopic. We can thus write $\mathbf{0}$ for the constant (t-)loop at 0 without specifying its domain.

Proposition 6.3. *$\langle \pi_1(G), \cdot, [\mathbf{0}] \rangle$ is a group, with $[\gamma] \cdot [\delta] := [\gamma \vee \delta]$.*

Proof. Definition 6.2 provides that for all t-paths $\gamma, \gamma', \delta, \delta'$, if $\gamma \sim_t \gamma', \delta \sim_t \delta'$, then $(\gamma \vee \delta) \sim_t (\gamma' \vee \delta')$, and therefore $\cdot$ is well-defined. Associativity is trivial since for all t-paths γ, δ, σ, $(\gamma \vee \delta) \vee \sigma = \gamma \vee (\delta \vee \sigma)$. Clearly, $[\mathbf{0}]$ is a left and right unit element. Finally, for $\gamma : [0,p] \to G$ a t-path, the class of $\gamma^*(t) := \gamma(p-t)$ is the left and right inverse $[\gamma]^{-1}$ of $[\gamma]$. Indeed, $(\gamma \vee \gamma^*) \sim_t \mathbf{0} : [0,2p] \to \{0\}$ is witnessed by the t-homotopy $F(t,s) : [0,2p] \times [0,p] \to G$, $F_t = \gamma_t \vee \gamma_t^*$, where $\gamma_t(u) : [0,p] \to G$ is a t-path with

$$\gamma_t(u) = \begin{cases} \gamma(u) & \text{if } 0 \leq u \leq t, \\ \gamma(t) & \text{if } t \leq u \leq p. \end{cases}$$

Replacing γ by γ^*, we get also $(\gamma^* \vee \gamma) \sim_t \mathbf{0}$. □

Definition 6.4 ([BO2])**.** We call $\pi_1(G) = \langle \pi_1(G), \cdot, [\mathbf{0}] \rangle$ the *o-minimal fundamental group of G*.

Note: We could instead define $\pi_1(G,v) := \mathbb{L}(G,v)/\sim_t$, for every $v \in G$, where $\mathbb{L}(G,v)$ is the set of all t-loops that start and end at v. As it turns out, this is not necessary, since G is t-connected and $\pi_1(G,v)$ is, up to definable isomorphism, independent of the choice of v (by identically applying the classical proof of the same fact, as in [Hat, Proposition 1.5], for example).

Definition 6.5 ([vdD], Chapter 8, (3.1))**.** Let $A \subseteq X \subseteq M^m$. We say that X *deformation retracts* to A if there is a homotopy $F(t,s) : X \times [0,r] \to X$ such that $F(X,0) = A$, $F_1 = \mathbf{1}_X$, and $\forall s \in [0,r]$, $F(\cdot,s)\restriction_A = \mathbf{1}_A$.

Lemma 6.6. *For every $r \in M$, the n-box $\mathcal{B}_0^n(r) = (-r,r)^n \subset M^n$ deformation retracts to $\{0\}$.*

Proof. Let $B_m := \mathcal{B}_0^m(r) = (-r,r)^m \subset M^m$, $m > 0$, and $B_0 = \{0\}$. By induction, it suffices to show that for $m > 0$, B_m deformation retracts to B_{m-1}. But this is witnessed by the following homotopy in M^m: $F(t,s) : B_m \times [0,r] \to B_m$, with

$$F\big((t_1,\dots,t_m),s\big) = \begin{cases} (t_1,\dots,t_m) & \text{if } |t_m| \le s, \\ (t_1,\dots,t_{m-1},s) & \text{if } t_m > s, \\ (t_1,\dots,t_{m-1},-s) & \text{if } t_m < -s. \end{cases}$$

□

Corollary 6.7. *Let $\gamma : [0,p] \to M^n$ be a loop with $\gamma(0) = 0$. Then, $\gamma \sim \mathbf{0} : [0,p] \to \{0\}$.*

Proof. Since $[0,p] \subset M$ is closed and bounded, $\mathrm{Im}(\gamma)$ is (closed and) bounded by [PeS, Corollary 2.4], and thus $\mathrm{Im}(\gamma) \subseteq \mathcal{B}_0(r) \subset M^n$, for some $r \in M$. By Lemma 6.6, there is a deformation retraction $F(t,s) : \mathcal{B}_0(r) \times [0,q] \to M^n$ of $\mathcal{B}_0(r)$ to $\{0\}$. It is then not hard to check that $G(t,s) := F\big(\gamma(t),s\big) : [0,p] \times [0,q] \to M^n$ is a homotopy between γ and $\mathbf{0} : [0,p] \to \{0\}$. □

We now proceed to show that $\pi_1(G) \cong L = \ker(\phi)$. Let us first prove a useful lemma about paths and t-paths.

Lemma 6.8. *(i) Let $\delta : [0,p] \to U$ be a path. Then there are some $h_1,\dots,h_m \in H$ with definable slopes (Definition 3.2) and, $\forall i \in \{1,\dots,m\}$, linear paths $h_i(t) \in H$ from 0 to h_i, such that $\delta(t) = \big(\delta(0) + h_1(t)\big) \vee \big(\delta(0) + h_1 + h_2(t)\big) \vee \dots \vee \big(\delta(0) + h_1 + \dots + h_{m-1} + h_m(t)\big)$.*

(ii) Let $\gamma : [0,p] \to G$ be a t-path starting at $c \in G$. Then there are some $h_1,\dots,h_m \in H$ with definable slopes and, $\forall i \in \{1,\dots,m\}$, linear paths $h_i(t) \in H$ from 0 to h_i, such that $\gamma(t) = \big(c \oplus h_1(t)\big) \vee \big(c \oplus h_1 \oplus h_2(t)\big) \vee \dots \vee \big(c \oplus h_1 \oplus \dots \oplus h_{m-1} \oplus h_m(t)\big)$.

Proof. (i) By Remark 3.3(ii), it suffices to show the statement for δ being linear. Let $\delta(p) \in H^k$ for some $k \in \mathbb{N}$. Then, easily, $\frac{\delta(p)}{k} \in H$, and δ is the concatenation of k linear paths $\delta \restriction [0, \frac{p}{k}]$.

(ii) Let $\gamma(t) : [0,p] \to G$ with $\gamma(0) = c \in G$ and $H_1 := \frac{1}{2}H$. By Lemma 4.28, finitely many $\oplus$-translates, $\{a_i \oplus H_1\}_{\{1 \le i \le m\}}$, $m \in \mathbb{N}$, of H_1 cover $\mathrm{Im}(\gamma)$. By o-minimality, and since H_1 is t-open, we can assume that there are $0 = t_0, t_1, \dots, t_m \in [0,p]$, such that $\forall i \in \{1,\dots,m-1\}$, $\gamma(t_i) \in (a_i \oplus H_1) \cap (a_{i+1} \oplus H_1)$, $\gamma(t_0) = c \in a_1 \oplus H_1$, $\gamma(p) \in a_m \oplus H_1$, and that for each $i \in \{0,\dots,m-1\}$,

(a) $\gamma^{i+1} := \gamma \restriction_{[t_i,t_{i+1}]}$ lies in $a_i \oplus H_1$,

(b) $\gamma^{i+1} \restriction_{(t_i,t_{i+1})}$ is linear, and

(c) γ does not jump at any $t \in (t_i,t_{i+1})$.

By (b), for all $i \in \{0,\dots,m-1\}$, there exists some linear path $h_{i+1} : [t_i,t_{i+1}] \to M^n$ such that $\forall t \in (t_i,t_{i+1})$, $\gamma^{i+1}(t) = \gamma(t_i) + h_{i+1}(t)$. We denote $h_{i+1} := h_{i+1}(t_{i+1}) \in M^n$.

We work by induction on i. Suppose that for some $i \in \{1,\dots,m-1\}$, $\gamma(t_i) = c \oplus h_1 \oplus \dots \oplus h_i$ and $h_1,\dots,h_i \in H$. We show that $\forall t \in (t_i,t_{i+1}]$, $\gamma(t) = c \oplus h_1 \oplus \dots \oplus h_i \oplus h_{i+1}(t)$ and $h_{i+1} \in H$. Let us assume that γ^{i+1} does not jump at t_i. The other case can be handled similarly. By (c), γ^{i+1} does not jump at any $t \in [t_i,t_{i+1})$.

By (a), $\forall t \in (t_i,t_{i+1})$, $\gamma(t_i) + h_{i+1}(t) \in a_i \oplus H_1$. Since also $\gamma(t_i) \in a_i \oplus H_1$, we have $\big(\gamma(t_i) + h_{i+1}(t)\big) \ominus \gamma(t_i) \in (a_i \oplus H_1) \ominus (a_i \oplus H_1) \subseteq H$. By Lemma 4.23(ii), we have $\forall t \in (t_i, t_{i+1})$, $\big(\gamma(t_i) + h_{i+1}(t)\big) \ominus \gamma(t_i) = \big(\gamma(t_i) \ominus \gamma(t_i)\big) + h_{i+1}(t) = h_{i+1}(t)$. This shows that

$$\forall t \in [t_i, t_{i+1}),\ \gamma(t) = \gamma(t_i) + h_{i+1}(t) = \gamma(t_i) \oplus h_{i+1}(t).$$

We thus have:

$$\gamma(t_{i+1}) = \lim_{t \to t_{i+1}^-} {}^t\gamma(t) = \lim_{t \to t_{i+1}^-} {}^t[\gamma(t_i) \oplus h_{i+1}(t)] = \gamma(t_i) \oplus h_{i+1}(t_{i+1}).$$

That $h_{i+1} \in H$ is then also clear, since $h_{i+1}(t_{i+1}) = \gamma(t_{i+1}) \ominus \gamma(t_i) \in (a_i \oplus H_1) \ominus (a_i \oplus H_1) \subseteq H$. □

Lemma 6.9. $\ker(\phi) = \{J_\gamma : \gamma$ *is a t-loop*$\}$.

Proof. $\subseteq$. This is just Lemma 4.33. For $x \in \ker(\phi)$ and γ as in that proof, we have $x = -J_\gamma = J_{\gamma^*}$.

$\supseteq$. Let $\gamma(t)$ be a t-loop starting and ending at $c \in G$, and $h_1, \dots, h_m \in H$ as in Lemma 6.8(ii). Since γ is a t-loop, we have: $c \oplus h_1 \oplus \dots \oplus h_m = c$, thus $h_1 \oplus \dots \oplus h_m = 0$. On the other hand, by Proposition 4.24, $c \oplus h_1 \oplus \dots \oplus h_m = c + \sum_{i=0}^m h_i + J_\gamma$, thus $J_\gamma = -\sum_{i=0}^m h_i$. Therefore, $\phi(J_\gamma) = \phi\Big(-\sum_{i=0}^m h_i\Big) = \ominus\phi\Big(\sum_{i=0}^m h_i\Big) = \ominus(h_1 \oplus \dots \oplus h_m) = 0$. □

For a t-path $\gamma : [0,p] \to G$ starting at c, we fix some h_i and $[t_{i-1}, t_i] \ni t \mapsto h_i(t) \in H$, $i \in \{1, \dots, m\}$, to be as in Lemma 6.8(ii).

Definition 6.10. Let $\gamma : [0,p] \to G$ be a t-path starting at $c \in G$. Let $d \in U$ such that $\phi(d) = c$. The *lifting of* γ *at* d is the following path $\hat\gamma_d : [0,p] \to U$,

$$\hat\gamma_d(t) = \big(d + h_1(t)\big) \vee \big(d + h_1 + h_2(t)\big) \vee \dots \vee \big(d + h_1 + \dots + h_{m-1} + h_m(t)\big).$$

Let γ as above be in addition a t-loop. By Proposition 4.24, $c = c \oplus h_1 \oplus \dots \oplus h_m = c + h_1 + \dots + h_m + J_\gamma$. It follows that

$$J_\gamma = 0 \Leftrightarrow h_1 + \dots + h_m = 0 \Leftrightarrow \hat\gamma_d \text{ is a loop in } U.$$

Lemma 6.11. *(i) For any t-path $\gamma : [0,p] \to G$ starting at c, and $d \in U$ such that $\phi(d) = c$, we have $\phi \circ \hat\gamma = \gamma$.*

(ii) For any path $\delta : [0,p] \to U$, $J_{\phi\circ\delta} = \phi\big(\delta(p)\big) - \phi\big(\delta(0)\big) - \big(\delta(p) - \delta(0)\big)$. In particular, for any loop δ in U, $J_{\phi\circ\delta} = 0$.

Proof. (i) Clear, since $\phi\big(d + h_1 + \dots + h_{i-1} + h_i(t)\big) = c \oplus h_1 \oplus \dots \oplus h_{i-1} \oplus h_i(t)$.

(ii) By Lemma 6.8(i), let $h_1, \dots, h_m \in H$ have definable slopes and, $\forall i \in \{1, \dots, m\}$, let $h_i(t) \in H$ be a linear path from 0 to h_i, such that $\delta(t) = \big(\delta(0) + h_1(t)\big) \vee \big(\delta(0) + h_1 + h_2(t)\big) \vee \dots \vee \big(\delta(0) + h_1 + \dots + h_{m-1} + h_m(t)\big)$. It is then $\delta = \hat\gamma_{\delta(0)}$, where $\gamma(t) = \big(c \oplus h_1(t)\big) \vee \big(c \oplus h_1 \oplus h_2(t)\big) \vee \dots \vee \big(c \oplus h_1 \oplus \dots \oplus h_{m-1} \oplus h_m(t)\big)$, with $c = \phi(\delta(0))$. By (i), $\phi \circ \delta = \gamma$, and thus Proposition 4.24 gives $c \oplus h_1 \oplus \dots \oplus h_m = c + \sum_{i=0}^m h_i + J_{\phi\circ\delta}$. Therefore, $J_{\phi\circ\delta} = (c \oplus h_1 \oplus \dots \oplus h_m) - c - \sum_{i=0}^m h_i = \phi\big(\delta(p)\big) - \phi\big(\delta(0)\big) - \big(\delta(p) - \delta(0)\big)$. □

Lemma 6.12. *For every $\gamma \in \mathbb{L}(G)$, $\gamma \sim_t \mathbf{0} \Leftrightarrow J_\gamma = 0$.*

Proof. ($\Leftarrow$). Let $\gamma \in \mathbb{L}(G)$ with $J_\gamma = 0$. Then $\hat\gamma_0$ is a loop in U, homotopic to $\mathbf{0}$ by Corollary 6.7. Since ϕ is t-continuous, the image of the homotopy under ϕ is a t-homotopy between γ and $\mathbf{0}$.

($\Rightarrow$). Assume now $\gamma \sim_t \mathbf{0}$, witnessed by $F(t,s) : [0,p] \times [0,r] \to G$, say $\gamma(t) = F_r(t) = F(t,r)$. Since $F(0,s) = 0 = F(p,s)$ for all s, the paths $\widehat{F(0,s)}_0$, $\widehat{F(p,s)}_0$ should equal $\mathbf{0}$. This means that for all s, $\widehat{(F_s)}_0$ is a loop in U. By Lemma 6.11(i), $J_\gamma = J_{\phi\circ\widehat{\gamma}}$, and by Lemma 6.11(ii), $J_{\phi\circ\widehat{(F_r)}_0} = 0$. It follows, $J_\gamma = J_{\phi\circ\widehat{\gamma}} = J_{\phi\circ\widehat{(F_r)}_0} = 0$. □

Proposition 6.13. $\pi_1(G) \cong \ker(\phi) = L$.

Proof. By Lemma 6.9, we have to show that the map $j : \pi_1(G) \ni [\gamma] \mapsto J_\gamma \in \{J_\gamma : \gamma \text{ is a } t\text{-loop}\}$ is a group isomorphism. Note: $\forall \gamma, \delta \in \mathbb{L}(G)$, $J_{\gamma\vee\delta} = J_\gamma + J_\delta$ and $J_{\gamma^*} = -J_\gamma$. Now, j is well-defined and one-to-one since for all $\gamma : [0,p] \to G$ and $\delta : [0,q] \to G$ in $\mathbb{L}(G)$,

$$[\gamma] = [\delta] \Leftrightarrow [\gamma] \cdot [\delta^*] = 0 \Leftrightarrow [\gamma \vee \delta^*] = 0 \Leftrightarrow J_{\gamma\vee\delta^*} = 0 \Leftrightarrow J_\gamma = J_\delta,$$

where the third equivalence is by Lemma 6.12. Trivially, j is onto, and it is a group homomorphism by the above note. □

Remark 6.14. The pair $\langle U, \phi\rangle$ can be considered as a universal covering space for G, in the sense that (i) there is a definable t-open covering $\{G_i\}$ of G such that every $\phi^{-1}(G_i)$ is a disjoint union of open sets in U, each of which is mapped by ϕ homeomorphically onto G_i, and (ii) U is 'definably' simply-connected. Indeed:

(i) Let $\{a_i \oplus H\}$ be a finite t-open covering of G by $\oplus$-translates of H. We show that for all i, $\phi^{-1}(a_i \oplus H) = \bigsqcup_{\phi(x)=a}(x + H)$ is a disjoint union of open sets in U. Let $x \neq y$ with $\phi(x) = \phi(y)$. We show $(x+H) \cap (y+H) = \emptyset$. If there were $h_1, h_2 \in H$ with $x + h_1 = y + h_2$, then $\phi(h_1 - h_2) = \phi(y - x) = 0$, and thus $\phi(h_1) = \phi(h_2)$. Since ϕ restricted to H is the identity, we have $h_1 = h_2$. Thus, $x = y$, a contradiction. It is also not hard to see that ϕ restricted to $x + H$ is a homeomorphism onto $\phi(x) \oplus H$.

(ii) U is easily definably path-connected, and, by Corollary 6.7, simply-connected.

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