Certificates for non negative polynomials over semialgebraic sets

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UNIVERSITAT DE BARCELONA

Barcelona's Computational Algebra

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- Joel Hurtado – UB
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“The” Question

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Given a polynomial

$$f(x_1, \ldots, x_n) \in \mathbb{R}/\mathbb{Q}[x_1, \ldots, x_n]$$

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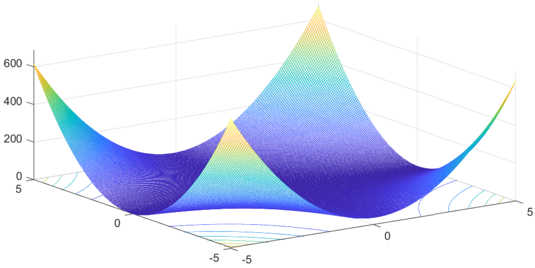
How can we verify/certify if $f \geq 0$?



Univariate case

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$$f(x) \geq 0 \; \forall x \in \mathbb{R} \iff$$

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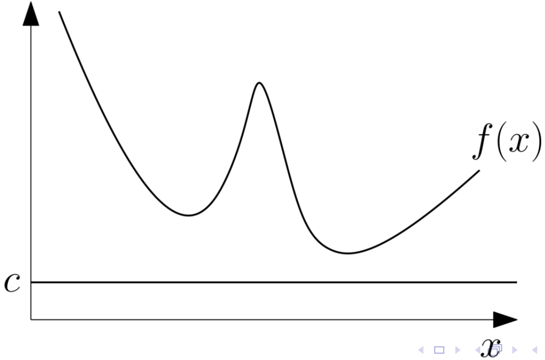
$$f(x) \geq 0 \ \forall x \in \mathbb{R} \iff f(x) = f_1(x)^2 + f_2(x)^2$$



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$$f(x) \geq 0 \ \forall x \in \mathbb{R} \iff$$
$$f(x) =$$
$$f_1(x)^2 + f_2(x)^2 + f_3(x)^2 + f_4(x)^2 + f_5(x)^2$$

Pourchet – 1971

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$$f(x_1, x_2) = 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3) \geq 0$$

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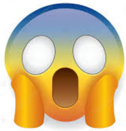

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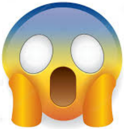

Negative solution to Hilbert's 17th Problem

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$$f(x_1,\ldots,x_n)\geq 0 \iff$$
$$\exists h, f_1,\ldots,f_k\in\mathbb{R}[x_1,\ldots,x_n],\ h\neq 0,$$
with
$$h^2\,f = f_1^2+\ldots+f_k^2$$

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$f(x_1, \ldots, x_n)$ is a sos $\iff$ $\exists B \in \mathbb{R}^{N \times N}$, $B^t = B$, positive semidefinite, such that

$$f = (1\, x_1\, \ldots x^\alpha \ldots) \cdot B \cdot (1\, x_1\, \ldots x^\alpha \ldots)^t$$

Testing s.o.s

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- Computing if a given semialgebraic set is empty or not (Powers)
- Solving a semidefinite linear problem (Laserre)

Reals versus racionals

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$$40x_0^4 + 8x_0^2x_1^2 + 32x_0^2x_1x_2 + 64x_0^2x_1x_3$$
$$+16x_0^2x_2^2 + 16x_0^2x_2x_3 + 32x_0^2x_3^2 + 2x_1^4$$
$$+8x_1^2x_2^2 + 8x_1^2x_2x_3 + 16x_1x_2x_3^2$$
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but cannot written as a sos with polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$

Bounds for Artin's representation

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$$\deg(p, f_1, \ldots, f_k) \leq 2^{2^{2^{d^{4^k}}}}$$

Lombardi-Perrucci-Roy 2000

The dictionary Algebra-Geometry

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$\mathbb{K}$ a field

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$$f|_{V(f_1, \ldots f_k)} = 0 \iff f^\ell \in \langle f_1, \ldots, f_k \rangle$$

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$$f^{2\ell} + g_1^2 + \ldots + g_r^2 \in \langle f_1, \ldots, f_k \rangle$$

Krivine–1964

Algebraic and semialgebraic sets

Algebraic and semialgebraic sets

$$V(f_1, \ldots, f_n)$$
$$=$$
$$\{x \in \mathbb{K}^n : \ f_i(x) = 0, \ 1 \leq i \leq k\}$$

Algebraic and semialgebraic sets

$$V(f_1, \ldots, f_n) = \{x \in \mathbb{K}^n : \; f_i(x) = 0, \, 1 \leq i \leq k\}$$

$$S(f_1, \ldots, f_n) = \{x \in \mathbb{K}^n : \; f_i(x) \geq 0, \, 1 \leq i \leq k\}$$

The preordering & Positivstellensatz

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$$PO(f_1, \ldots, f_k)$$
$$= \{\sum_{e\in\{0,1\}^k} s_e f_1^{e_1} \ldots f_k^{e_k}\},$$
$$s_e \in SOS(\mathbb{R}[x])$$

The preordering & Positivstellensatz

$$PO(f_1, \ldots, f_k) = \{\sum_{e \in \{0,1\}^k} s_e f_1^{e_1} \ldots f_k^{e_k}\},\ s_e \in SOS(\mathbb{R}[x])$$

Positivstellensatz (Stengle 1974)

$$f|_{S(f_1,\ldots,f_k)} > 0 \iff s\,f = 1 + t,\ s, t \in PO(f_1, \ldots, f_k)$$

Nichtnegativstellensatz

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(Stengle 1974)

$$f|_{S(f_1,\ldots,f_k)} \geq 0 \iff s\,f = f^{2\ell} + t,\ \ell \in \mathbb{N}, s, t \in PO(f_1, \ldots, f_k)$$

Nichtnegativstellensatz

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Computational aspects are hard!

The quadratic module (easier)

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$$M(f_1, \ldots, f_k) :=$$
$$\{g_0 + g_1 f_1 + \ldots + g_k f_k\}$$
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$$M(f_1, \ldots, f_k) \subset PO(f_1, \ldots, f_k)$$

$f \in M(f_1, \ldots, f_k)$ can be easily tested

Quadratic modules and positivity

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When do we have

$$f|_{S(f_1,\ldots,f_k)} \geq 0 \Rightarrow f \in M(s_1,\ldots,s_k)?$$

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$M(f_1,\ldots,f_k)$ is called **archimedean**

if $N - x_1^2 - \ldots - x_n^2 \in M(f_1,\ldots,f_k)$

for some $N > 0$

Easy to check

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$M(f_1, \ldots, f_k)$ archimedean

$\Rightarrow S(f_1, \ldots, f_k) \subset \mathbb{R}^n$ compact

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$$M(f_1,\dots,f_k) \text{ archimedean}$$
$$\Rightarrow S(f_1,\dots,f_k) \subset \mathbb{R}^n \text{ compact}$$

Putinar's Positivstellensatz

If $M(f_1,\dots,f_k)$ is archimedean and $f|_{S(f_1,\dots,f_n)} > 0$ then $f \in M(f_1,\dots,f_k)$

Cases of applications/extensions

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- Polytopes Powers 2004

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- Polytopes Powers 2004
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- Zero-dimensional ideals
  Krick-Mourrain-Szanto 2020 –
  Baldi-Krick-Mourrain 2024

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Given $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n]$

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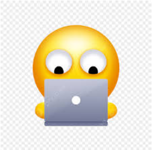
Given $f_1, \ldots, f_k \in \mathbb{R}[x_1, \ldots, x_n]$, how to detect if $M(f_1, \ldots, f_k)$ is archimedean?



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Poster by Joel Hurtado in the next Young RSME meeting 2025

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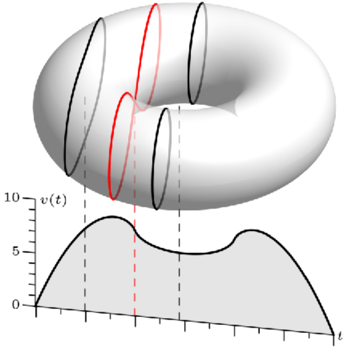

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Thanks!

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