Toric Euler-Jacobi vanishing theorem and zeros at infinity

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ELSEVIER

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Explicit formulas for the multivariate resultant

Carlos D'Andrea, Alicia Dickenstein [1 2]

Today

Today

Yet another joint work with Alicia Dickenstein!

Univariate interpolation

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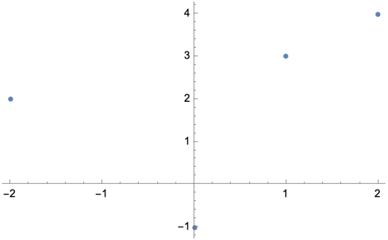
$$(x_1, y_1), \ldots, (x_N, y_N) \subset \mathbb{K}^2 \quad x_i \neq x_j \quad i \neq j$$



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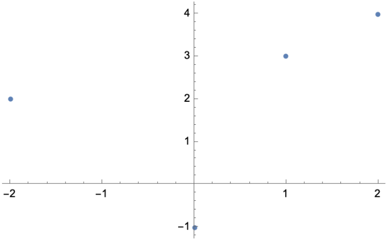
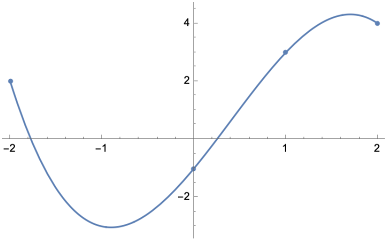


find the polynomial of smallest degree passing through these points

Univariate interpolation



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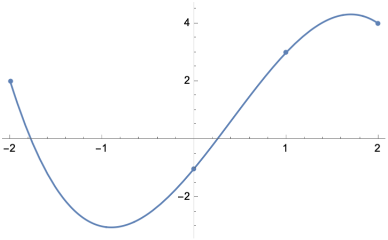

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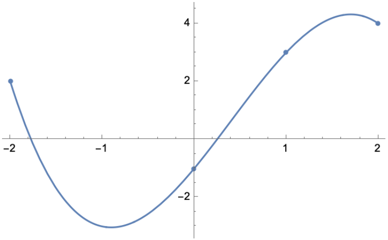

$$f(x) = \prod_{i=1}^{N}(x - x_i)$$

$$p(x) := \sum_{i=1}^{N} y_i \, \frac{\prod_{j\neq i}(x - x_j)}{f'(x_i)}$$

Interpolating polynomial

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$$\iff \; 0 = \sum_{i=1}^{N} \frac{y_i}{f'(x_i)}$$

$$= \sum_{i=1}^{N} \frac{p(x_i)}{f'(x_i)} =: \mathrm{Res}_g\left(\frac{p(x)}{f(x)}\right)$$

Moreover...

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Any $p(x) \in \mathbb{K}[x]$ can be expressed modulo $f(x)$ as

$$p(x) = \sum_{i=0}^{N-2} \lambda_i x^i + \frac{\mathrm{Res}_g\left(\frac{p(x)}{f(x)}\right)}{N} f'(x)$$

with $\lambda_0, \ldots, \lambda_{N-2} \in \mathbb{K}$

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Can be defined algebraically

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Euler-Jacobi Vanishing Theorem

If $\deg(p(x)) < \deg(f(x)) - 1$ then $\mathrm{Res}_g\left(\frac{p(x)}{f(x)}\right) = 0$

Consequences

$$p(x) = h_0 + h_1 x + \ldots + h_{d-1} x^{d-1} \bmod f(x)$$

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$$\mathrm{Res}_g\left(\frac{x^i\, p(x)}{f(x)}\right) = 0, \;\; i = 0, 1, \ldots, d-1$$

Multivariate residues

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$f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ a zero-dimensional system $(\overline{\mathbb{K}} = \mathbb{K}, \mathrm{char}(\mathbb{K}) = 0)$

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- $\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\right)$ is computable

Moreover...

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$$p \in \langle f_1, \ldots, f_n \rangle \iff$$
$$\mathrm{Res}_g\left(\frac{b_j\, p}{f_1 \ldots f_n}\right) = 0 \; j = 1, \ldots, \ell$$

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$\Longrightarrow$ effective membership tests

Dickenstein-Sessa MEGA 1990

Euler-Jacobi vanishing Theorem

Euler-Jacobi vanishing Theorem

THEOREMATA NOVA ALGEBRAICA CIRCA SYSTEMA DUARUM AEQUATIONUM INTER DUAS VARIABILES PROPOSITARUM.

1.

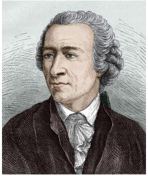
E theorematis, quae in elementis algebraicis traduntur, vix extat aliud magis utile in aequationibus maxime diversis, quam notum illud:

„Designante X functionem ipsius x rationalem integram, fieri

$$\Sigma\left(\frac{U}{\frac{dX}{dx}}\right)=0,$$

„si quidem extendatur summa ad omnes radices x aequationis $X=0$,

Jacobi MEGA 1835



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- $\deg(p) < d_1 + \ldots + d_n - n$

$$\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\right) = 0$$

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If $\deg_a(V(f_1, \ldots, f_n)) = d_1 \ldots d_n$ any $p \in \mathbb{K}[x_1, \ldots, x_n]$ can be expressed mod $\langle f_1, \ldots, f_n \rangle$

Multivariate Interpolation

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as

$$p_0 + \frac{\mathrm{Res}_g\left(\frac{h}{f_1 \ldots f_n}\right)}{d_1 \ldots d_n} J_f$$

with $\deg(p_0) < d_1 + \ldots + d_n - n$

Multivariate Interpolation II

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$p(\xi_j) = y_j$ with

$$\deg(p) < d_1 + \ldots + d_n - n \iff$$

$$\sum_{j=1}^{d} \frac{y_j}{J_f(\xi_j)} = \mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\right) = 0$$

A “negative” consequence

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If $\deg_a(V(f_1,\ldots,f_n)) = d_1 \ldots d_n$

A "negative" consequence

If $\deg_a(V(f_1, \ldots, f_n)) = d_1 \ldots d_n$ you will never find a $\mathbb{K}$-basis of $\mathbb{K}[x_1, \ldots, x_n]/\langle f_1, \ldots, f_n \rangle$ made of polynomials of degree $< d_1 + \ldots + d_n - n$

Residues in the torus

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zero-dimensional

Local and global residue are defined as usual:

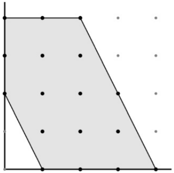
$$\mathrm{Res}_g\left(\frac{h}{f_1 \ldots f_n}\right) := \sum_{\xi \in V(f)} \mathrm{Res}_\xi\left(\frac{h}{f_1 \ldots f_n}\right)$$

Computing residues

Computing residues

Easy in the generic case

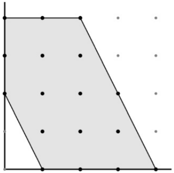
$$\deg_t(V(f_1,\ldots,f_n)) = MV(N(f_1),\ldots,N(f_n)) =: \mathrm{m}$$



Computing residues

Easy in the generic case

$$\deg_t(V(f_1,\ldots,f_n)) = MV(N(f_1),\ldots,N(f_n)) =: \mathrm{m}$$



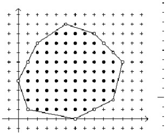
Cattani-Dickenstein-Sturmfels MEGA 1992

Cattani-Dickenstein MEGA 1994

Euler-Jacobi in the torus

Euler-Jacobi in the torus

If $\deg_t(V(f_1,\ldots,f_n)) = m$ and $N(t_1\ldots t_n\,p) \subset (N(f_1)+\ldots+N(f_n))^\circ$



Euler-Jacobi in the torus

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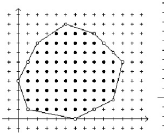

$$\Longrightarrow \boxed{\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\right) = 0}$$

Khovanskii 78

Cattani-Dickenstein MEGA 1994

Sparse Interpolation

If $\deg_t(V(f_1,\ldots,f_n)) = \mathrm{m}$

Sparse Interpolation

If $\deg_t(V(f_1, \ldots, f_n)) = m$ any $p \in \mathbb{K}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ mod $\langle f_1, \ldots, f_n \rangle$

Sparse Interpolation

If $\deg_t(V(f_1,\ldots,f_n)) = m$ any

$p \in \mathbb{K}[t_1^{\pm 1},\ldots,t_n^{\pm 1}]$ mod $\langle f_1,\ldots,f_n\rangle$

$$= p_0 + \frac{\mathrm{Res}_g\left(\frac{p}{f_1\ldots f_n}\right)}{\deg_t(V(f))} J_f$$

with

$$N(t_1\ldots t_n\, p_0) \subset (N(f_1)+\ldots+N(f_n))^\circ$$

Soprunov 07

Sparse Interpolation II

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$$N(t_1 \ldots t_n\, p) \subset$$

$$(N(f_1) + \ldots + N(f_n))^\circ \iff$$

$$\sum_{j=1}^{\mathrm{m}} \frac{y_j}{J_f(\xi_j)} = \mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\right) = 0$$

Soprunov 07

How do you prove these results?

Compactifications!

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homogenizations

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$$J_{x_0, f_1^h, \ldots, f_n^h} \notin \langle x_0, f_1^h \ldots, f_n^h \rangle \iff$$
$$\dim \left(\mathbb{K}[x_0, \ldots, x_n] / \langle x_0, f_1^h, \ldots, f_n^h \rangle \right)_{d_1 + \ldots + d_n - n} = 1$$

Homogeneous residues

Homogeneous residues

The map

$$\begin{array}{ccc} \left(\mathbb{K}[x_0,\ldots,x_n]/\langle x_0,f_1^h,\ldots,f_n^h\rangle\right)_{d_1+\ldots+d_n-n} & \to & \mathbb{K} \\ J_{x_0,f_1^h,\ldots,f_n^h} & \mapsto & d_1\ldots d_n \end{array}$$

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$x_0 \mid p^h$

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If $\deg(p) < d_1+\ldots+d_n-n$

$x_0 \,|\, p^h \implies$ Euler-Jacobi

Sparse compatifications

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homogenizations to the **Cox ring**

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X is a toric variety adapted to the $N(f_i)'s$

Toric residues

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There is a trace map

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giving the global residue

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Cox 96

Cattani, Cox, Dickenstein 97

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very special cases

Cox 96

Cattani, Cox, Dickenstein 97

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Cox 96

Cattani, Cox, Dickenstein 97

"Combinatorial" jacobian

Cattani, Cox, Dickenstein 97

D, Khetan MEGA 2003

Khetan, Soprunov 05

Codimension one?

$$\dim\left(\mathbb{K}[x_1,\ldots,x_{n+r}]/\langle x^\alpha, f_1^h,\ldots,f_n^h\rangle\right)_\rho = 1$$

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Combinatorial characterization given in

Cox, Dickenstein 05

Today: MEGA 2024

(joint with Alicia Dickenstein)

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Is it true that

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(joint with Alicia Dickenstein)

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$$\text{Euler-Jacobi} \iff \deg_t(V(f)) = \mathrm{m}?$$

Today: MEGA 2024

(joint with Alicia Dickenstein)

Is it true that

$$\text{Euler-Jacobi} \iff \deg_t(V(f)) = \text{m?}$$

$$J_{f_0^h,\dots,f_n^h} \in \langle f_0^h, \dots, f_n^h \rangle \iff$$

$$V(f_h^0, \dots, f_n^h) \neq \emptyset?$$

Euler-Jacobi

Euler-Jacobi

Is it true that if $\deg_t(V(f)) < \mathrm{m}$?
then $\exists p : \; N(t_1 \ldots t_n\, p) \subset$
$(N(f_1) + \ldots + N(f_n))^\circ$
such that $\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\right) \neq 0$?

First obstruction: codimension!

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$$f_1 = 2 - t_1 + t_1^2 + t_2 + 2t_1t_2 + t_2^2$$
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$$N(f_1) = 2\Delta_2 \times \{0\}$$
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First obstruction: codimension!

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fails the codimension 1 property

Only obstruction (so far):

D-Dickenstein MEGA 2024

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D-Dickenstein MEGA 2024

Assume that $N(f_1), \ldots, N(f_n)$ is **"completable"**.

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D-Dickenstein MEGA 2024

Assume that $N(f_1), \ldots, N(f_n)$ is **"completable"**. If $\deg_t(V(\boldsymbol{f})) <$ m and $\dim(V_X(f^h)) = 0$

Only obstruction (so far):

D-Dickenstein MEGA 2024

Assume that $N(f_1), \ldots, N(f_n)$ is **"completable"**. If $\deg_t(V(f)) < \mathrm{m}$ and $\dim(V_X(f^h)) = 0$ then $\exists p : N(t_1 \ldots t_n p) \subset (N(f_1) + \ldots + N(f_n))^{\circ}$ such that $\mathrm{Res}_g\left(\frac{p}{f_1 \ldots f_n}\right) \neq 0$

Second question

D-Dickenstein MEGA 2024

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D-Dickenstein MEGA 2024

If generically

$$\dim\left(\mathbb{K}[x_1,\ldots,x_{n+r}]/\langle F_0^h, F_1^h,\ldots,F_n^h\rangle\right)_\rho = 1$$

Second question

D-Dickenstein MEGA 2024

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$\dim\left(\mathbb{K}[x_1,\ldots,x_{n+r}]/\langle F_0^h,F_1^h,\ldots,F_n^h\rangle\right)_\rho = 1$ but

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D-Dickenstein MEGA 2024

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D-Dickenstein MEGA 2024

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and $\dim(V_X(f_1^h,\ldots,f_n^h)) = 0$

then $J_{f_0^h,\ldots,f_n^h} \in \langle f_0^h,\ldots,f_n^h\rangle$

A "good" consequence

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If $\deg_a(V(f_1, \ldots, f_n)) < \mathrm{m}$

A "good" consequence

If $\deg_a(V(f_1, \ldots, f_n)) < \mathrm{m}$
there is a simplicial cone $\sigma \in \Sigma_X$
where you can get a basis of the
quotient ring $\mathbb{K}[X_\sigma]/\langle f_1^a, \ldots, f_n^a \rangle$
with support contained in
$(N(f_1) + \ldots + N(f_n))^\circ$

Tools

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- Geometry of toric varieties

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- Local study of toric / in the torus residues

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- Geometry of toric varieties
- Local study of toric / in the torus residues
- Useful properties of multiplicities of points in affine toric varieties (Bender-Telen 22)

Last but not least

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ISSAC 2025

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July 28th to August 1st 2025

Guanajuato, Mexico

https://www.issac-conference.org/2025/

Have a great MEGA!

UNIVERSITAT DE BARCELONA

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