Sparse Nullstellensatz, Resultants and Determinants of Complexes

Carlos D'Andrea & Gabriela Jeronimo (Buenos Aires)

El Escorial, June 24 2024

UNIVERSITAT DE BARCELONA

Hilbert's (weak) Nullstellensatz

Hilbert's (weak) Nullstellensatz

1893

Hilbert's (weak) Nullstellensatz

1893

$$f_1, \ldots, f_k \in R_K := K[t_1, \ldots, t_n]$$

Hilbert's (weak) Nullstellensatz

1893

$$f_1, \ldots, f_k \in R_K := K[t_1, \ldots, t_n]$$
$$V_{\overline{K}^n}(f_1, \ldots, f_k) = \emptyset$$

Hilbert's (weak) Nullstellensatz

1893

$$f_1, \ldots, f_k \in R_K := K[t_1, \ldots, t_n]$$
$$V_{\overline{K}^n}(f_1, \ldots, f_k) = \emptyset$$
$$\Longleftrightarrow$$
$$\exists g_1, \ldots, g_k \in R_K : \ 1 = \sum_{j=1}^k g_j f_j$$

Effective Nullstellensatz

Effective Nullstellensatz

Given a bound for $\deg(g_j)$

Effective Nullstellensatz

Given a bound for $\deg(g_j)$

$1 = \sum_{j=1}^{k} g_j f_j$ turns into a linear algebra problem

Effective Nullstellensatz

Given a bound for $\deg(g_j)$

$1 = \sum_{j=1}^{k} g_j f_j$ turns into a linear algebra problem

$$\deg(g_j) \leq (k\, d_f)^{2^n}$$

Grette Herman – 1926

State of the art

$$\deg(g_j) \leq {d_f}^k$$

State of the art

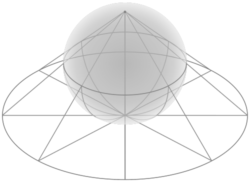
$$\deg(g_j) \leq {d_f}^k$$

Sharp bound, but....

State of the art

$$\deg(g_j) \leq {d_f}^k$$

Sharp bound, but....



Geometry can help improving the bounds

Projective space

$$\overline{K}^n \subset \mathbb{P}^n$$

Projective space

$$\overline{K}^n \subset \mathbb{P}^n$$

$$V_{\overline{K}^n}(f_1, \ldots, f_k) \subset V_{\mathbb{P}^n}(f_1, \ldots, f_k)$$

Projective space

$$\overline{K}^n \subset \mathbb{P}^n$$

$$V_{\overline{K}^n}(f_1, \ldots, f_k) \subset V_{\mathbb{P}^n}(f_1, \ldots, f_k)$$

Theorem (Macaulay 1902)

If $V_{\mathbb{P}^n}(f_1, \ldots, f_k) = \emptyset$ then

Projective space

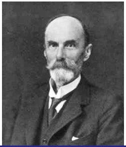
$$\overline{K}^n \subset \mathbb{P}^n$$

$$V_{\overline{K}^n}(f_1, \ldots, f_k) \subset V_{\mathbb{P}^n}(f_1, \ldots, f_k)$$

Theorem (Macaulay 1902)

If $V_{\mathbb{P}^n}(f_1, \ldots, f_k) = \emptyset$ then

$$\deg(g_j) \leq n\, d_f - n$$



Sparse polynomials

Sparse polynomials

$\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$ finite sets

$$f_i := \sum_{b \in \mathcal{A}_i} f_{i,b} t^b \in K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

Sparse polynomials

$\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$ finite sets

$$f_i := \sum_{b \in \mathcal{A}_i} f_{i,b} t^b \in K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

$$f_{i,b} \in K^\times := K \setminus \{0\}$$

Sparse polynomials

$\mathcal{A}_1, \ldots, \mathcal{A}_k \subset \mathbb{Z}^n$ finite sets

$$f_i := \sum_{b \in \mathcal{A}_i} f_{i,b} t^b \in K[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$$

$$f_{i,b} \in K^\times := K \setminus \{0\}$$
$$\mathbb{T}^n := (K^\times)^n$$

Sparse weak Nullstellensatz

Sparse weak Nullstellensatz

$$V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$$

Sparse weak Nullstellensatz

$$V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$$

$$\Longleftrightarrow$$

$$\exists g_1, \ldots, g_k \in K[\mathrm{t}^{\pm 1}] : \; 1 = \sum_{j=1}^{k} g_j f_j$$

Sparse weak Nullstellensatz

$$V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$$

$$\Longleftrightarrow$$

$$\exists g_1, \ldots, g_k \in K[\mathrm{t}^{\pm 1}] : \; 1 = \sum_{j=1}^{k} g_j f_j$$

Effective sparse Nullstellensatz?

Effective Sparse Nullstellensatz

Theorem (Sombra 99)

If $V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$, then

Effective Sparse Nullstellensatz

Theorem (Sombra 99)

If $V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$, then
$N(g_j) + N(f_j) \subset -a + n^{2n+3} v^2 N_{1,\ldots,k}$

Effective Sparse Nullstellensatz

Theorem (Sombra 99)

If $V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$, then

$$N(g_j) + N(f_j) \subset -a + n^{2n+3} v^2 N_{1,\ldots,k}$$

where

Effective Sparse Nullstellensatz

Theorem (Sombra 99)

If $V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$, then

$$N(g_j) + N(f_j) \subset -a + n^{2n+3} v^2 N_{1,\ldots,k}$$

where

- $N(f) \subset \mathbb{R}^n$ is the Newton polytope of f

Effective Sparse Nullstellensatz

Theorem (Sombra 99)

If $V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$, then

$$N(g_j) + N(f_j) \subset -a + n^{2n+3} v^2 N_{1,\ldots,k}$$

where

- $N(f) \subset \mathbb{R}^n$ is the Newton polytope of f
- $N_{1,\ldots,k} \subset \mathbb{R}^n$ is the convex hull of $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$

Effective Sparse Nullstellensatz

Theorem (Sombra 99)

If $V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$, then

$N(g_j) + N(f_j) \subset -a + n^{2n+3} v^2 N_{1,\ldots,k}$

where

- $N(f) \subset \mathbb{R}^n$ is the Newton polytope of f
- $N_{1,\ldots,k} \subset \mathbb{R}^n$ is the convex hull of $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$
- $v = n! \operatorname{vol}(N_{1,\ldots,k})$

Effective Sparse Nullstellensatz

Theorem (Sombra 99)

If $V_{\mathbb{T}^n}(f_1, \ldots, f_k) = \emptyset$, then

$$N(g_j) + N(f_j) \subset -a + n^{2n+3} v^2 N_{1,\ldots,k}$$

where

- $N(f) \subset \mathbb{R}^n$ is the Newton polytope of f
- $N_{1,\ldots,k} \subset \mathbb{R}^n$ is the convex hull of $\mathcal{A}_1 \cup \ldots \cup \mathcal{A}_k$
- $v = n! \operatorname{vol}(N_{1,\ldots,k})$
- $a \in \mathbb{Z}^n$

Can Geometry help us?

Can Geometry help us?

$$\mathbb{T}^n \subset X_{\mathcal{A}}$$

Can Geometry help us?

$$\mathbb{T}^n \subset X_{\mathcal{A}}$$

$X_{\mathcal{A}}$ is the toric variety associated to $\mathcal{A}_1, \ldots, \mathcal{A}_k$

Can Geometry help us?

$$\mathbb{T}^n \subset X_{\mathcal{A}}$$

$X_{\mathcal{A}}$ is the toric variety associated to

$$\mathcal{A}_1, \ldots, \mathcal{A}_k$$

$$V_{\mathbb{T}^n}(f_1, \ldots, f_k) \subset V_{X_{\mathcal{A}}}(f_1, \ldots, f_k)$$

Nullstellensatz over $X_{\mathcal{A}}$

Nullstellensatz over $X_{\mathcal{A}}$

Theorem (Tuitman 11)

If $V_{X_{\mathcal{A}}}(f_1, \ldots, f_k) = \emptyset$ then for any $a \in \mathbb{Z}^n \cap \sum_{i=1}^k N(f_i)$,

Nullstellensatz over $X_{\mathcal{A}}$

Theorem (Tuitman 11)

If $V_{X_{\mathcal{A}}}(f_1, \ldots, f_k) = \emptyset$ then for any $a \in \mathbb{Z}^n \cap \sum_{i=1}^{k} N(f_i)$,

$$N(g_j) + N(f_j) \subset -a + \sum_{i=1}^{k} N(f_i)$$

Our refinement

Theorem (D-Jeronimo 24)

If $V_{X_{\mathcal{A}}}(f_1, \ldots, f_k) = \emptyset$, for generic small $\delta \in \mathbb{Q}^n$:

Our refinement

Theorem (D-Jeronimo 24)

If $V_{X_{\mathcal{A}}}(f_1, \ldots, f_k) = \emptyset$, for generic small $\delta \in \mathbb{Q}^n$:

$$N(g_j) + N(f_j) \subset -a \boxed{-\delta} + \sum_{i=1}^{k} N(f_i).$$

Our refinement

Theorem (D-Jeronimo 24)

If $V_{X_{\mathcal{A}}}(f_1, \ldots, f_k) = \emptyset$, for generic small $\delta \in \mathbb{Q}^n$:

$N(g_j) + N(f_j) \subset -a \boxed{-\delta} + \sum_{i=1}^{k} N(f_i)$.

The bound is sharp.

How do you prove these results?

$f_1, \ldots, f_k \in S_K$ of degrees $d_1, \ldots, d_k$

How do you prove these results?

$f_1, \ldots, f_k \in S_K$ of degrees $d_1, \ldots, d_k$

$F_1, \ldots, F_k \in \tilde{R}_K := K[t_0, \ldots, t_n]$,

their homogenizations

How do you prove these results?

$f_1, \ldots, f_k \in S_K$ of degrees $d_1, \ldots, d_k$

$F_1, \ldots, F_k \in \tilde{R}_K := K[t_0, \ldots, t_n]$,

their homogenizations

$$0 \to \tilde{R}_K^{\binom{k}{k}} \xrightarrow{\Psi_k} \tilde{R}_K^{\binom{k}{k-1}} \xrightarrow{\Psi_{k-1}} \ldots \xrightarrow{\Psi_2} \tilde{R}_K^{\binom{k}{1}} \xrightarrow{\Psi_1} \tilde{R}_K^{\binom{k}{0}} \to 0$$

How do you prove these results?

$f_1, \ldots, f_k \in S_K$ of degrees $d_1, \ldots, d_k$

$F_1, \ldots, F_k \in \tilde{R}_K := K[t_0, \ldots, t_n]$,

their homogenizations

$$0 \to \tilde{R}_K^{\binom{k}{k}} \xrightarrow{\Psi_k} \tilde{R}_K^{\binom{k}{k-1}} \xrightarrow{\Psi_{k-1}} \ldots \xrightarrow{\Psi_2} \tilde{R}_K^{\binom{k}{1}} \xrightarrow{\Psi_1} \tilde{R}_K^{\binom{k}{0}} \to 0$$

the Koszul Complex of the F_i's

Properties

- $Kos(\tilde{R}_K, F_1, \ldots, F_k)$ is graded

Properties

- $Kos(\tilde{R}_K, F_1, \ldots, F_k)$ is graded
- $Kos(\tilde{R}_K, F_1, \ldots, F_k)_{d_1+\ldots+d_k-n}$ is exact

Properties

- $Kos(\tilde{R}_K, F_1, \ldots, F_k)$ is graded
- $Kos(\tilde{R}_K, F_1, \ldots, F_k)_{d_1+\ldots+d_k-n}$ is exact $\iff$ the last map is onto

Properties

- $Kos(\tilde{R}_K, F_1, \ldots, F_k)$ is graded
- $Kos(\tilde{R}_K, F_1, \ldots, F_k)_{d_1+\ldots+d_k-n}$ is exact $\iff$ the last map is onto $\iff$ $V_{\mathbb{P}^n}(F_1, \ldots, F_k) = \emptyset$

Properties

- $Kos(\tilde{R}_K, F_1, \ldots, F_k)$ is graded
- $Kos(\tilde{R}_K, F_1, \ldots, F_k)_{d_1+\ldots+d_k-n}$ is exact $\iff$ the last map is onto $\iff$ $V_{\mathbb{P}^n}(F_1, \ldots, F_k) = \emptyset$
- $\implies$ bounds for the degrees for the Nullstellensatz over $\mathbb{P}^n$

Sparse Koszul

$f_1, \ldots, f_k \in K[\mathrm{t}^{\pm 1}]$ with supports

$\mathcal{A}_1, \ldots, \mathcal{A}_k$

Sparse Koszul

$f_1, \ldots, f_k \in K[\mathrm{t}^{\pm 1}]$ with supports

$\mathcal{A}_1, \ldots, \mathcal{A}_k$

$F_1, \ldots, F_k \in S_K := K[x_1, \ldots, x_{n+r}]$

their "homogenizations" of "degrees"

$\alpha_1, \ldots, \alpha_k \in A_{n-1}(X_{\mathcal{A}})$

Sparse Koszul

$f_1, \ldots, f_k \in K[t^{\pm 1}]$ with supports

$\mathcal{A}_1, \ldots, \mathcal{A}_k$

$F_1, \ldots, F_k \in S_K := K[x_1, \ldots, x_{n+r}]$

their "homogenizations" of "degrees"

$\alpha_1, \ldots, \alpha_k \in A_{n-1}(X_{\mathcal{A}})$

$$0 \to \ S_K^{\binom{k}{k}} \stackrel{\Psi_k}{\to} S_K^{\binom{k}{k-1}} \stackrel{\Psi_{k-1}}{\to} \ldots \stackrel{\Psi_2}{\to} S_K^{\binom{k}{1}} \stackrel{\Psi_1}{\to} S_K^{\binom{k}{0}} \to 0$$

Properties

- $Kos(S_K, F_1, \ldots, F_k)$ is $A_{n-1}(X_{\mathcal{A}})$-graded

Properties

- $Kos(S_K, F_1, \ldots, F_k)$ is $A_{n-1}(X_{\mathcal{A}})$-graded
- $\alpha_\delta \in A_{n-1}(X_{\mathcal{A}})$ "shift" by $\delta \in \mathbb{Q}^n$

Properties

- $Kos(S_K, F_1, \ldots, F_k)$ is $A_{n-1}(X_{\mathcal{A}})$-graded
- $\alpha_\delta \in A_{n-1}(X_{\mathcal{A}})$ "shift" by $\delta \in \mathbb{Q}^n$
- $Kos(S_K, F_1, \ldots, F_k)_{\alpha_1+\ldots+\alpha_k-\alpha_\delta}$ is exact

Properties

- $Kos(S_K, F_1, \ldots, F_k)$ is $A_{n-1}(X_{\mathcal{A}})$-graded
- $\alpha_\delta \in A_{n-1}(X_{\mathcal{A}})$ "shift" by $\delta \in \mathbb{Q}^n$
- $Kos(S_K, F_1, \ldots, F_k)_{\alpha_1+\ldots+\alpha_k-\alpha_\delta}$ is exact $\iff$ the last map is onto

Properties

- $Kos(S_K, F_1, \ldots, F_k)$ is $A_{n-1}(X_{\mathcal{A}})$-graded
- $\alpha_\delta \in A_{n-1}(X_{\mathcal{A}})$ "shift" by $\delta \in \mathbb{Q}^n$
- $Kos(S_K, F_1, \ldots, F_k)_{\alpha_1+\ldots+\alpha_k-\alpha_\delta}$ is exact $\iff$ the last map is onto $\iff$ $V_{X_{\mathcal{A}}}(F_1, \ldots, F_k) = \emptyset$

Properties

- $Kos(S_K, F_1, \ldots, F_k)$ is $A_{n-1}(X_{\mathcal{A}})$-graded
- $\alpha_\delta \in A_{n-1}(X_{\mathcal{A}})$ "shift" by $\delta \in \mathbb{Q}^n$
- $Kos(S_K, F_1, \ldots, F_k)_{\alpha_1+\ldots+\alpha_k-\alpha_\delta}$ is exact $\iff$ the last map is onto $\iff$ $V_{X_{\mathcal{A}}}(F_1, \ldots, F_k) = \emptyset$
- $\implies$ bounds for the Nullstellensatz

Application: sparse resultants

Application: sparse resultants

$$f_i^g := \sum_{b \in \mathcal{A}_i} c_{i,b} t^b, \ 1 \leq i \leq n+1$$

Application: sparse resultants

$$f_i^g := \sum_{b \in \mathcal{A}_i} c_{i,b} t^b, \; 1 \leq i \leq n+1$$

$c_{i,b}$ indeterminates

Application: sparse resultants

$$f_i^g := \sum_{b \in \mathcal{A}_i} c_{i,b} t^b, \; 1 \leq i \leq n+1$$

$c_{i,b}$ indeterminates

$$\mathbb{K} := \mathbb{Q}(c_{i,b})$$

Application: sparse resultants

$$f_i^g := \sum_{b\in\mathcal{A}_i} c_{i,b}t^b,\ 1 \le i \le n+1$$

$c_{i,b}$ indeterminates

$$\mathbb{K} := \mathbb{Q}(c_{i,b})$$

$S_{\mathbb{K}} := \mathbb{K}[x_1, \ldots, x_{n+r}]$, the Cox ring of $X_{\mathcal{A}}$

Sparse resultants

Sparse resultants

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} \in \mathbb{Z}[c_{i,b}]$$

Sparse resultants

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} \in \mathbb{Z}[c_{i,b}]$$

if it is not identically 1

Sparse resultants

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} \in \mathbb{Z}[c_{i,b}]$$

if it is not identically 1 it vanishes under any specialization $f_i^g \mapsto f_i \in K[t^{\pm 1}]$

Sparse resultants

$$\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} \in \mathbb{Z}[c_{i,b}]$$

if it is not identically 1 it vanishes under any specialization $f_i^g \mapsto f_i \in K[t^{\pm 1}] \iff V_{X_{\mathcal{A}}}(F_1,\ldots,F_{n+1}) \neq \emptyset$

Sparse resultants

$\text{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} \in \mathbb{Z}[c_{i,b}]$ **if it is not identically** 1 it vanishes under any specialization $f_i^g \mapsto f_i \in K[t^{\pm 1}] \iff V_{X_\mathcal{A}}(F_1,\ldots,F_{n+1}) \neq \emptyset$

(Recall $\text{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}} = \text{Elim}^p_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$)

Generic Koszul complex

Generic Koszul complex

$$Kos(S_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_1+\ldots+\alpha_{n+1}-\alpha_\delta}$$

is exact

Generic Koszul complex

$$Kos(S_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_1+\ldots+\alpha_{n+1}-\alpha_\delta}$$

is exact

$$Kos(S_{\mathbb{Q}[c_{i,b}]}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_1+\ldots+\alpha_{n+1}-\alpha_\delta}$$

is "generically" exact

Generic Koszul complex

$$Kos(S_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_1+\ldots+\alpha_{n+1}-\alpha_\delta}$$

is exact

$$Kos(S_{\mathbb{Q}[c_{i,b}]}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_1+\ldots+\alpha_{n+1}-\alpha_\delta}$$

is "generically" exact

We can compute the **determinant** of this complex (wrt monomial bases)

Determinants of exact complexes

Determinants of exact complexes

$$0 \to V_r \xrightarrow{d_r} V_{r-1} \xrightarrow{d_{r-1}} \ldots \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \to 0$$

exact complex of finite dimensional vector spaces.

Determinants of exact complexes

$$0 \to V_r \stackrel{d_r}{\to} V_{r-1} \stackrel{d_{r-1}}{\to} \ldots \stackrel{d_2}{\to} V_1 \stackrel{d_1}{\to} V_0 \to 0$$

exact complex of finite dimensional vector spaces.

$$V_i = E_{i+1} \bigoplus E_i, \quad d_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

$b_i : E_i \to E_i$ invertible

Determinants of exact complexes

$$0 \to V_r \xrightarrow{d_r} V_{r-1} \xrightarrow{d_{r-1}} \ldots \xrightarrow{d_2} V_1 \xrightarrow{d_1} V_0 \to 0$$

exact complex of finite dimensional vector spaces.

$$V_i = E_{i+1} \bigoplus E_i, \quad d_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$$

$b_i : E_i \to E_i$ invertible

$$\boxed{\det(V_*) = \prod \det(b_i)^{(-1)^{i+1}}}$$

Resultant=determinant of complex

Resultant=determinant of complex

Theorem (D-Jeronimo 24)

- The determinant of $Kos(S_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_1+\ldots+\alpha_{n+1}-\alpha_\delta}$ is equal to $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$

Resultant=determinant of complex

Theorem (D-Jeronimo 24)

- The determinant of $Kos(S_{\mathbb{K}}, F_1^g, \ldots, F_{n+1}^g)_{\alpha_1+\ldots+\alpha_{n+1}-\alpha_\delta}$ is equal to $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$
- $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$ is equal to the gcd of the maximal minors of the last nontrivial map of the complex

Consequences/Remarks

Consequences/Remarks

- Any maximal minor of Canny-Emiris matrices is a multiple of $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$

Consequences/Remarks

- Any maximal minor of Canny-Emiris matrices is a multiple of $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$(conjectured in [DJS23])

Consequences/Remarks

- Any maximal minor of Canny-Emiris matrices is a multiple of $\mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}$(conjectured in [DJS23])
- **It is not true that** $Kos(S_K, F_1, \ldots, F_{n+1})_{\alpha_1+\ldots+\alpha_{n+1}-\alpha_\delta}$ is exact $\iff \mathrm{Res}_{\mathcal{A}_1,\ldots,\mathcal{A}_{n+1}}(f_1, \ldots, f_{n+1}) \neq 0$

More general setup

More general setup

- Generalized Canny-Emiris matrices use an extra lattice polytope $Q \subset \mathbb{R}^n$

More general setup

- Generalized Canny-Emiris matrices use an extra lattice polytope $Q \subset \mathbb{R}^n$
- Must "enlarge" the toric variety: $X_{\mathcal{A},Q} \to X_{\mathcal{A}}$

More general setup

- Generalized Canny-Emiris matrices use an extra lattice polytope $Q \subset \mathbb{R}^n$
- Must "enlarge" the toric variety: $X_{\mathcal{A},Q} \to X_{\mathcal{A}}$
- We can work in $X_{\mathcal{A},Q}$ with more general degrees $\alpha_Q + \alpha_1 + \ldots + \alpha_{n+1} - \alpha_\delta$ making our result also valid for all the generalized Canny-Emiris matrices

Thanks!

cdandrea@ub.edu
http://www.ub.edu/arcades/cdandrea.html