The use of higher order syzygies in the implicitization of rational parametrizations

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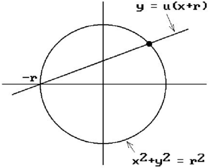
UNIVERSITAT DE BARCELONA

ARCADES

Setup: Homogeneous Coordinates

$$\begin{array}{ccc} \mathbb{K} & \dashrightarrow & \mathbb{K}^2 \\ t & \longmapsto & \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right) \end{array}$$



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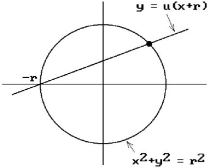
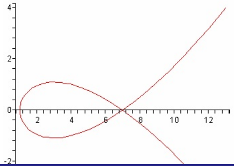


$$\begin{array}{cccc} \phi : & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ & (t_0 : t_1) & \longmapsto & (t_0^2 + t_1^2 : t_0^2 - t_1^2 : 2t_0t_1) \end{array}$$

Rational Plane Parametrizations

$$\begin{array}{rccc} \phi: & \mathbb{P}^1 & \to & \mathbb{P}^2 \\ & (t_0 : t_1) & \mapsto & (a(t_0,t_1) : b(t_0,t_1) : c(t_0,t_1)) \end{array}$$

- $a, b, c \in \mathbb{K}[T_0, T_1]$, homogeneous of the same degree $d \geq 1$
- $\gcd(a, b, c) = 1$



Rational Curves in the plane

The image of ϕ is a **rational plane curve**

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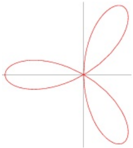

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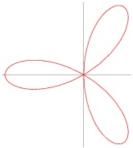

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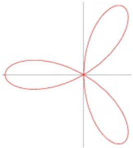

- It has degree d if ϕ is "generically" injective
- It has genus 0, which means the maximal number of multiple points $\frac{(d-1)(d-2)}{2}$
- Computing its implicit equation is relatively easy from the input ϕ

Implicit Equations

$$X_2 a(\underline{T}) - X_0 c(\underline{T}) \;=\; X_2 T_0^2 - 2X_0 T_0 T_1 + X_2 T_1^2$$

$$X_2 b(\underline{T}) - X_1 c(\underline{T}) \;=\; X_2 T_0^2 - 2X_1 T_0 T_1 - X_2 T_1^2$$

$$\mathrm{Res}_{\underline{T}}(X_2 \cdot a(\underline{T}) - X_0 \cdot c(\underline{T}), X_2 \cdot b(\underline{T}) - X_1 \cdot c(\underline{T})) \\ = \\ \det \begin{pmatrix} X_2 & -2X_0 & X_2 & 0 \\ 0 & X_2 & -2X_0 & X_2 \\ X_2 & -2X_1 & -X_2 & 0 \\ 0 & X_2 & -2X_1 & -X_2 \end{pmatrix} = -4X_2^2(X_0^2 - X_1^2 - X_2^2)$$

How small can the matrix be?

$$\mathcal{L}_{1,1}(\underline{T},\underline{X}) = X_2 \, T_0 - (X_0 + X_1) \, T_1$$
$$\mathcal{L}'_{1,1}(\underline{T},\underline{X}) = (-X_0 + X_1) \, T_0 + X_2 \, T_1$$

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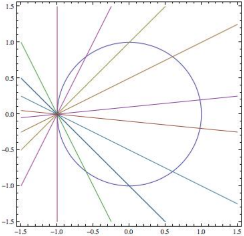
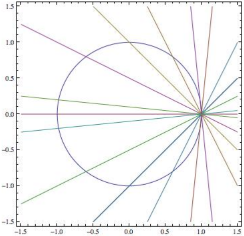


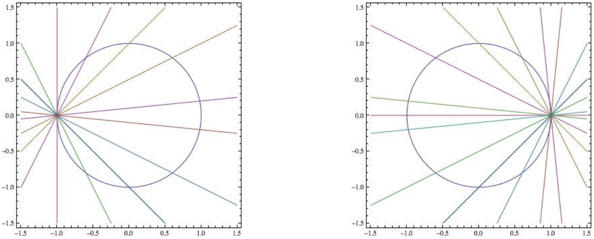


$$\det\begin{pmatrix} X_2 & -X_0-X_1 \\ -X_0+X_1 & X_2 \end{pmatrix} = X_1^2+X_2^2-X_0^2$$

Hilbert's Syzygy Theorem

There exist $\mu \leq \frac{d}{2}$ and two other parametrizations $\varphi_\mu(t_0, t_1)$, $\psi_{d-\mu}(t_0, t_1)$ of degrees μ, $d-\mu$ such that

$$\boxed{\phi(t_0, t_1) = \varphi_\mu(t_0, t_1) \wedge \psi_{d-\mu}(t_0, t_1)}$$



For the unit circle...

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$$\begin{aligned} \varphi_1(t_0 : t_1) &= (-t_1 : -t_1 : t_0) \\ \psi_1(t_0 : t_1) &= (-t_0 : t_0 : t_1) \end{aligned}$$

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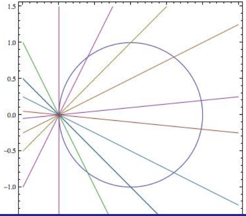
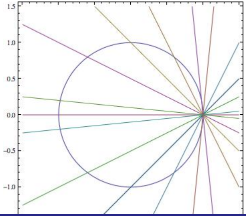
$$\begin{vmatrix} \mathbf{e}_0 & \mathbf{e}_1 & \mathbf{e}_2 \\ -t_1 & -t_1 & t_0 \\ -t_0 & t_0 & t_1 \end{vmatrix} = \left(-t_0^2 - t_1^2, t_1^2 - t_0^2, -2t_0t_1\right)$$





Algebraic Version

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The homogeneous ideal
$I = (a(\underline{T}), b(\underline{T}), c(\underline{T})) \subset \mathbb{K}[T_0, T_1]$ has a
Hilbert-Burch resolution of the type

$$0 \to \mathbb{K}[\underline{T}]^2 \xrightarrow{(\varphi_\mu, \psi_{d-\mu})^{\mathbf{t}}} \mathbb{K}[\underline{T}]^3 \xrightarrow{(a,b,c)} \mathbb{K}[\underline{T}]$$

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A μ-basis of the parametrization is a basis of $\mathrm{Syz}(I)$ as a $\mathbb{K}[\underline{T}]$-module

Why do we care about μ-bases?

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Implicit equation

$$=$$

$$\mathrm{Res}_{\underline{T}}\big(\varphi_\mu(\underline{T}),\ \varphi_{d-\mu}(\underline{T})\big)$$

Computing μ-bases

A moving line

$$\mathcal{L}(T_0, T_1, X_0, X_1, X_2) = v_0(\underline{T})X_0 + v_1(\underline{T})X_1 + v_2(\underline{T})X_2$$

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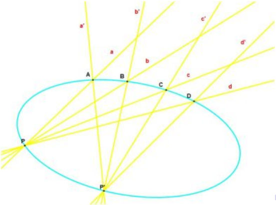
follows the parametrization iff

$$\mathcal{L}(T_0, T_1, a(\underline{T}), b(\underline{T}), c(\underline{T})) = 0$$



In our example...

$$
\begin{array}{rcl}
\mathcal{L}_1(\underline{T},\underline{X}) & = & -2T_0^2T_1X_0 + 0X_1 + (T_0^3 + T_0T_1^2)X_2 \\
\mathcal{L}_2(\underline{T},\underline{X}) & = & -2T_0T_1^2X_0 + 0X_1 + (T_0^2T_1 + T_1^3)X_2 \\
\mathcal{L}_3(\underline{T},\underline{X}) & = & 0X_0 - 2T_0^2T_1X_1 + (T_0^3 - T_0T_1^2)X_2 \\
\mathcal{L}_4(\underline{T},\underline{X}) & = & 0X_0 - 2T_0T_1^2X_1 + (T_0^2T_1 - T_1^3)X_2
\end{array}
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\end{array}
$$

$$
\begin{pmatrix}
X_2 & -2X_0 & X_2 & 0 \\
0 & X_2 & -2X_0 & X_2 \\
X_2 & -2X_1 & -X_2 & 0 \\
0 & X_2 & -2X_1 & -X_2
\end{pmatrix}
$$

In general

The determinant of a "matrix of moving lines" is a multiple of the implicit equation

$$\begin{pmatrix} L_{11}(\underline{X}) & L_{12}(\underline{X}) & \dots & L_{1k}(\underline{X}) \\ L_{21}(\underline{X}) & L_{22}(\underline{X}) & \dots & L_{2k}(\underline{X}) \\ \vdots & \vdots & \dots & \vdots \\ L_{k1}(\underline{X}) & L_{k2}(\underline{X}) & \dots & L_{kk}(\underline{X}) \end{pmatrix}$$

Moving conics, moving cubics,...

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$$\mathcal{O}(\underline{T})X_0^2 + \mathcal{P}(\underline{T})X_0X_1 + \mathcal{Q}(\underline{T})X_0X_2 + \mathcal{R}(\underline{T})X_1^2 + \mathcal{S}(\underline{T})X_1X_2 + \mathcal{T}(\underline{T})X_2^2 \in \mathbb{K}[\underline{T}, \underline{X}]$$

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is a **moving conic** following the parametrization if

$$\mathcal{O}(\underline{T})a(\underline{T})^2 + \mathcal{P}(\underline{T})a(\underline{T})b(\underline{T}) + \mathcal{Q}(\underline{T})a(\underline{T})c(\underline{T}) + \mathcal{R}(\underline{T})b(\underline{T})^2 + \mathcal{S}(\underline{T})b(\underline{T})c(\underline{T}) + \mathcal{T}(\underline{T})c(\underline{T})^2 = 0$$

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the more **singular** the curve, the **simpler** the description of the determinant

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The implicit equation of a quartic can be computed as a 2×2 determinant.
If the curve has a triple point, then one row is linear and the other is cubic.
Otherwise, both rows are quadratic.

A quartic with a triple point

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$$\phi(t_0, t_1) = (t_0^4 - t_1^4 : -t_0^2 t_1^2 : t_0 t_1^3)$$
$$F(X_0, X_1, X_2) = X_2^4 - X_1^4 - X_0 X_1 X_2^2$$

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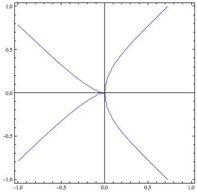

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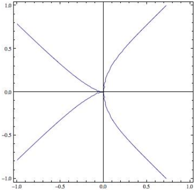

$$\mathcal{L}_{1,1}(\underline{T}, \underline{X}) = T_0 X_2 + T_1 X_1$$
$$\mathcal{L}_{1,3}(\underline{T}, \underline{X}) = T_0(X_1^3 + X_0 X_2^2) + T_1 X_2^3$$

$$\boxed{\begin{pmatrix} X_2 & X_1 \\ X_1^3 + X_0 X_2^2 & X_2^3 \end{pmatrix}}$$

A quartic without triple points

$$\phi(t_0 : t_1) = (t_0^4 : 6t_0^2t_1^2 - 4t_1^4 : 4t_0^3t_1 - 4t_0t_1^3)$$
$$F(\underline{X}) = X_2^4 + 4X_0X_1^3 + 2X_0X_1X_2^2 - 16X_0^2X_1^2 - 6X_0^2X_2^2 + 16X_0^3X_1$$

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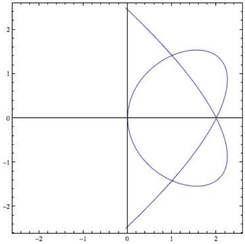

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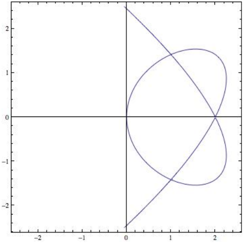

$$\mathcal{L}_{1,2}(\underline{T}, \underline{X}) = T_0(X_1X_2 - X_0X_2) + T_1(-X_2^2 - 2X_0X_1 + 4X_0^2)$$
$$\tilde{\mathcal{L}}_{1,2}(\underline{T}, \underline{X}) = T_0(X_1^2 + \tfrac{1}{2}X_2^2 - 2X_0X_1) + T_1(X_0X_2 - X_1X_2)$$

Very concentrated singularities

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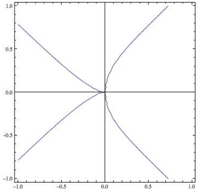

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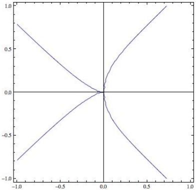

If the curve has a point of multiplicity $d-1$
the implicit equation is always a 2×2 determinant

$$\begin{vmatrix} \mathcal{L}_{1,1}(\underline{X}) & \mathcal{L}'_{1,1}(\underline{X}) \\ \mathcal{L}_{1,d-1}(\underline{X}) & \mathcal{L}'_{1,d-1}(\underline{X}) \end{vmatrix}$$

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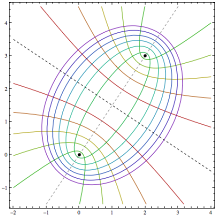
which moving lines?
which moving conics?
which moving cubics?
. . .



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Cox, D. Theoret. Comput. Sci. 392 (2008)
The moving curve ideal and the Rees algebra

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The moving curve ideal and the Rees algebra

$\mathcal{K}_\phi :=$ {Moving curves following ϕ} $=$
homogeneous elements in the kernel of

$$\begin{array}{ccc} \mathbb{K}[T_0, T_1, X_0, X_1, X_2] & \to & \mathbb{K}[T_0, T_1, s] \\ T_i & \mapsto & T_i \\ X_0 & \mapsto & a(\underline{T})s \\ X_1 & \mapsto & b(\underline{T})s \\ X_2 & \mapsto & c(\underline{T})s \end{array}$$

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"The ideal of moving curves following ϕ"

Method of moving curves revisited

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Method of moving curves revisited

The implicit equation should be obtained as the determinant of a matrix with

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The more singular the curve, the "simpler" the description of $\mathcal{K}_\phi$

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Compute a minimal system of generators of $\mathcal{K}_\phi$

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Compute a minimal system of generators of $\mathcal{K}_\phi$ for **any** ϕ

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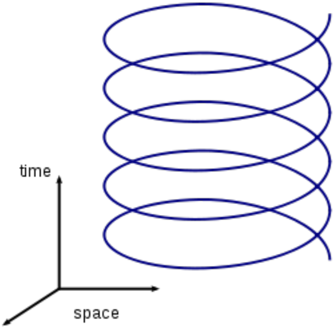
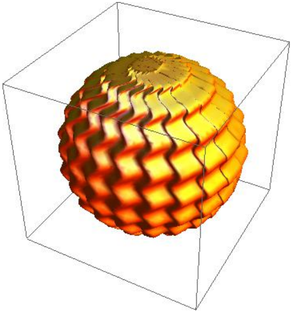
Known for

- $\mu = 1$ (Hong-Simis-Vasconcelos, Cox-Hoffmann-Wang, Busé, Cortadellas-**D**)
- $\mu = 2$ (Busé, Cortadellas-**D**, Kustin-Polini-Ulrich)
- $\left(\mathcal{K}_\phi\right)_{(1,2)} \neq 0$ (Cortadellas-**D**)
- Monomial Parametrizations (Cortadellas-**D**)

Only curves in the plane?





Rational Surfaces

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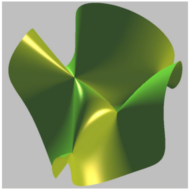

Rational Surfaces

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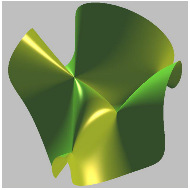

There are base points!

Implicitization via

- Resultants Macaulay, Dixon, Gelfand-Kapranov-Zelevinskii, ...

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Contrast:

- The module of moving planes is not free
- There is a concept of μ-basis given by Chen-Cox-Liu
  Not easy to compute (bounds on the degree by Cid Ruiz)

Some results

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Implicitization

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- Quadratic and cubic surfaces (Chen-Shen-Deng)

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Rees Algebras

Some results

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Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)

Some results

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- Quadratic and cubic surfaces (Chen-Shen-Deng)
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- ...

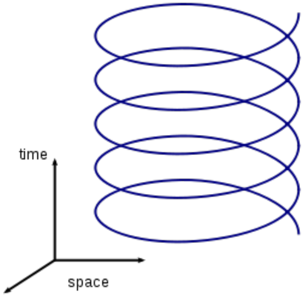
Rees Algebras

- "Monoid" Surfaces (Cortadellas - **D**)
- *de Jonquières* surfaces (Hassanzadeh- Simis)

Similar Results for

Spatial curves

$$\phi_C : \quad \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$$
$$\underline{t} = (t_0 : t_1) \longmapsto \big(a(\underline{t}) : b(\underline{t}) : c(\underline{t}) : d(\underline{t})\big)$$



Thanks!