Resultants modulo p

Carlos D'Andrea

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Computational Algebra,
Algebraic Geometry and Applications
A conference in honor of
Alicia Dickenstein
Buenos Aires, Argentina, August 1-3, 2016
Invited Speakers:
Carolina Araujo - Rio de Janeiro
Laurent Busé - Nice
Eduardo Cattani - Amherst
David Cox - Amherst
Fernando Cukierman - Buenos Aires
Sandra Di Rocco - Stockholm
Gabriela Jeronimo - Buenos Aires
Teresa Krick - Buenos Aires
Reinhard Laubenbacher - Connecticut
Laura Matusevich - College Station
Roberto Miatello - Córdoba
Bernard Mourrain - Nice
Marie-Françoise Roy - Rennes
Juan Sabia - Buenos Aires
Aron Simis - Pernambuco
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Univariate Resultants

$$\begin{cases} f_1 = a_{10}{x_0}^{d_1} + a_{11}{x_0}^{d_1-1}x_1 + \ldots + a_{1d_1}{x_1}^{d_1} \\ f_2 = a_{20}{x_0}^{d_2} + a_{21}{x_0}^{d_2-1}x_1 + \ldots + a_{2d_2}{x_1}^{d_2} \end{cases}$$

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$$\mathrm{Res}(f_1, f_2) = \det \begin{pmatrix} a_{10} & a_{11} & \ldots & a_{1d_1} & 0 & \ldots & 0 \\ 0 & a_{10} & \ldots & a_{1d_1-1} & a_{1d_1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{10} & \ldots & \ldots & a_{1d_1} \\ a_{20} & a_{21} & \ldots & a_{2d_2} & 0 & \ldots & 0 \\ 0 & a_{20} & \ldots & a_{2d_2-1} & a_{2d_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \ldots & \ldots & \ddots & \vdots \\ 0 & 0 & \ldots & a_{20} & \ldots & \ldots & a_{2d_2} \end{pmatrix}$$

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$$\boxed{p^{\deg(\gcd(f_1 \bmod p, f_2 \bmod p))} | \mathrm{Res}(f_1, f_2)}$$

(Gomez-Gutierrez-Ibeas-Sevilla 2009)

This fact has been used!

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bounding the cardinality of the reduction mod p of lengths of orbits of pairs of univariate dynamical systems

Igor Shparlinski's Question

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How general is this?

Vanishing of Resultantes modulo p

(Busé-D-Sombra 2016)

$$\begin{cases} f_1 = \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0}\ldots {x_n}^{\alpha_n} \\ f_2 = \sum_{\alpha_0+\ldots+\alpha_n=d_2} a_{2,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0}\ldots {x_n}^{\alpha_n} \\ \vdots \\ f_{n+1} = \sum_{\alpha_0+\ldots+\alpha_n=d_{n+1}} a_{n+1,\alpha_0,\ldots,\alpha_n} {x_0}^{\alpha_0}\ldots {x_n}^{\alpha_n} \end{cases}$$

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$$\boxed{f_1,\ldots,f_{n+1}\mathbb{Z}[\underline{x}] \implies \mathrm{Res}(f_1,\ldots,f_{n+1}) \in \mathbb{Z}}$$

Known case

If $d_1 = d_2 = \ldots = d_{n+1} = 1$, then

$$\mathrm{Res}(f_1, \ldots, f_{n+1}) = \det\left(a_{ij}\right)_{1\leq i,j\leq n+1}$$

A non trivial example

$$f_0 = a_{00}x_0 + a_{01}x_1 + a_{02}x_2$$
$$f_1 = a_{10}x_0 + a_{11}x_1 + a_{12}x_2$$
$$f_2 = a_{20}{x_0}^2 + a_{21}x_0x_1 + a_{22}x_0x_2 + a_{23}{x_1}^2 + a_{24}x_1x_2 + a_{25}{x_2}^2$$

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$$\begin{aligned}
\mathrm{Res}(f_0, f_1, f_2) = {} & a_{00}^2 a_{11}^2 a_{25} - a_{00}^2 a_{11} a_{12} a_{24} + a_{00}^2 a_{12}^2 a_{23} \\
& -2a_{00} a_{01} a_{10} a_{11} a_{25} + a_{00} a_{01} a_{10} a_{12} a_{24} + a_{00} a_{01} a_{11} a_{12} a_{22} \\
& -a_{00} a_{01} a_{12}^2 a_{21} + a_{00} a_{02} a_{10} a_{11} a_{24} - 2a_{00} a_{02} a_{10} a_{12} a_{23} \\
& -a_{00} a_{02} a_{11}^2 a_{22} + a_{00} a_{02} a_{11} a_{12} a_{21} + a_{01}^2 a_{10}^2 a_{25} \\
& -a_{01}^2 a_{10} a_{12} a_{22} + a_{01}^2 a_{12}^2 a_{20} - a_{01} a_{02} a_{10}^2 a_{24} \\
& +a_{01} a_{02} a_{10} a_{11} a_{22} + a_{01} a_{02} a_{10} a_{12} a_{21} - 2a_{01} a_{02} a_{11} a_{12} a_{20} \\
& +a_{02}^2 a_{10}^2 a_{23} - a_{02}^2 a_{10} a_{11} a_{21} + a_{02}^2 a_{11}^2 a_{20}
\end{aligned}$$

Properties of $\mathrm{Res}(f_1, \ldots, f_{n+1})$

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- It is irreducible
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- It is invariant under linear changes of coordinates

Geometric Properties

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- $\mathrm{Res}(f_1, \ldots, f_{n+1}) = 0 \iff$ $\exists \xi \in \mathbb{P}^n$ such that $f_1(\xi) = \ldots = f_{n+1}(\xi) = 0$

Geometric Properties

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- Poisson Formula:

$$\mathrm{Res}(f_1, \ldots, f_{n+1}) = \mathrm{Res}(f_1^0, \ldots, f_n^0)^{d_{n+1}} \prod_{\xi \in V(f_1^1, \ldots, f_n^1)} f_{n+1}(\xi)$$

Resolution of systems of polynomials

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can be used to compute the coordinates of the (finite) roots of the system

$$f_1 = 0, \ldots, f_n = 0$$

Computation

$$\mathcal{R}(f_0, f_1, f_2) = \det \begin{bmatrix} -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 \\ 0 & -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 \\ -c_4 & -c_5 & -c_8 & 0 & 0 & 0 & a_1 & 0 & a_3 & 0 \\ -c_1 & -c_3 & -c_7 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ -c_0 & -c_2 & -c_6 & 0 & 0 & 0 & 0 & a_0 & 0 & a_2 \\ 0 & 0 & 0 & -c_4 & -c_5 & -c_8 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & -c_1 & -c_3 & -c_7 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -c_0 & -c_2 & -c_6 & 0 & b_0 & 0 & b_2 \end{bmatrix}$$

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$$\boxed{p^{N_p}\,|\,\mathrm{Res}(f_1,\ldots,f_{n+1})}$$

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The factorization of the resultant actually bounds the (finite) zeroes modulo p for all prime p !

The Cantabrian Theorem revisited

$$\deg(\gcd(f_1 \bmod p, f_2 \bmod p))$$

$$=$$

$$N_p$$

$$=$$

$$\deg(V_p(f_1, f_2))$$

(Gomez-Gutierrez-Ibeas-Sevilla 2009)

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- the “gap” in the bound can be large
- Not a clear “algorithm” for $\dim\big(V_p(f_1,\ldots,f_{n+1})\big) > 0$

Idea of our proof

- "Remove" all the zeroes from the infinite

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- Apply Poisson formula
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- Compute the dimension of the Nullspace of the determinantal matrix

Generalizations and Extensions

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- One could get a refinement of the exponent by taking into account the valuation mod p of the roots (Smirnov's Theorem)
- Slight generalization to *sparse resultants* under stronger hypothesis
- The result holds for any domain, for instance polynomials with coefficients in $R[y_1, \ldots, y_l]$

Applications

- Finding points in varieties modulo p (Shparlinski)

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- The "Generalized Characteristic Polynomial" revisited! (Mourrain)
- "Extraneous factors" in the Computation of the "Salmon Polynomial" (Busé-Chardin-D-Sombra-Weimann)

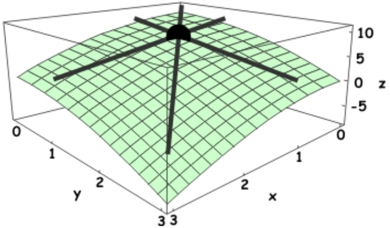
Computation of the Salmon Polynomial

(Busé-Chardin-D-Sombra-Weimann)

$$\mathbb{Z} \leftrightarrow \mathbb{C}[x,y,z,h]/\langle f(x,y,z,h)\rangle$$

$$p \leftrightarrow \qquad h$$

Salmon's polynomial



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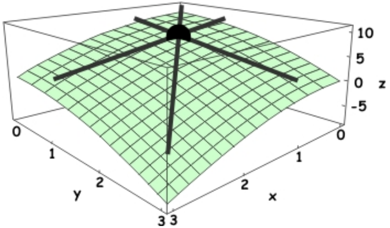

A point b in a surface $S \subset \mathbb{C}^3$ is called *flex* (or inflection) of S if there exists a line passing through b having contact order at least 3 with S

Theorem (Salmon, 1862)

If $S = V(f(x,y,z)) \subset \mathbb{C}^3$ of degree d, and not ruled, there is $F_f(x,y,z) \in \mathbb{C}[x,y,z]$ of degree $\leq 11d - 24$ such that

$$\boxed{Flex(S) = V(f(x,y,z), F_f(x,y,z))}$$

Computing $F_f(x, y, z)$

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$$f((x, y, z) + t(u, v, w)) =$$
$$f(x, y, z) + t\, f_1(x, y, z; u, v, w) +$$
$$t^2\, f_2(x, y, z; u, v, w) +$$
$$t^3\, f_3(x, y, z; u, v, w) + \mathcal{O}(t^4)$$

Computing $F_f(x, y, z)$

The "candidate" for $F_f(x, y, z)$ should be the resultant in (u, v, w) of

- $f_1(x, y, z; u, v, w)$
- $f_2(x, y, z; u, v, w)$
- $f_3(x, y, z; u, v, w)$

Janos Kollár (Adv. Math. 2015)

"I get a polynomial of degree $11d - 18$. Salmon claims that in fact the degree should be $11d - 24$. I have not checked this"

Terence Tao (blog, 2014)

"The original proof of the Cayley-Salmon theorem, dating back to at least 1915, is not easily accessible and not written in modern language"

Our Result

(Busé-Chardin-D-Sombra-Weimann)

Modulo $f(x, y, z, h)$, if we set $h = 0$ we get a nontrivial solution of the system of multiplicity 6

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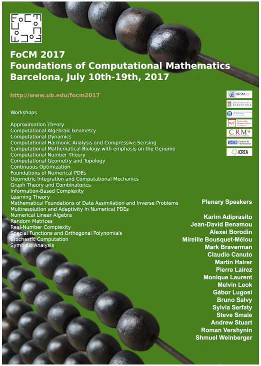
$$\mathrm{Res}(f_1,f_2,f_3) = h^6 \cdot F_f(x,y,z) \bmod f(x,y,z,h)$$



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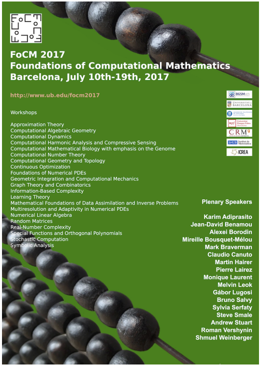

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Thanks!