GEOMETRIC SPARSE RESULTANTS

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1. Introduction

Sparse resultants are widely used in polynomial equation solving, a fact that has sparked a lot of interest in their computational and applied aspects, see for instance [CE00, EM99, Stu02, D'A02, JKSS04, CLO05, DE05, JMSW09]. They have also been studied from a more theoretical point of view because of their connections with combinatorics, toric geometry, residue theory, and hypergeometric functions [GKZ94, Stu94, CDS98, Kho99, CDS01, Est10].

Sparse elimination theory focuses on ideals and varieties defined by Laurent polynomials with given supports, in the sense that the exponents in their monomial expansion are *a priori* determined. The classical approach to this theory consists in regarding such Laurent polynomials as global sections of line bundles on a suitable projective toric variety. Using this interpretation, sparse elimination theory can be reduced to projective elimination theory. In particular, sparse resultants can be studied *via* the Chow form of this projective toric variety as it is done in [PS93, GKZ94, Stu94]. This approach works correctly when all considered line bundles are very ample, but need to be properly defined in general.

In these notes we will introduce informally elimination theory from both an algebraic and geometric point of view, and then focus in sparse elimination theory to exhibit some results obtained recently in [DS15] to show the versatility and applicability of this tool to polynomial system solving.

2. Elimination Theory: Algebra and Geometry

Elimination Theory lies at the core of Algebra and its applications, as everyone who has attended a class of elementary Mathematics has found it in many of its several aspects, including the most elementary following example.

Problem 2.1. Let $\mathbb{K}$ be a field, and a_{00}, a_{00}, $a_{10}, a_{11} \in \mathbb{K}$, find "the condition" under which the system of equations

$$\left\{ \begin{array}{rcl} a_{00}x_0 + a_{01}x_1 & = & 0 \\ a_{10}x_0 + a_{11}x_1 & = & 0 \end{array} \right. \tag{2.1}$$

has a solution in $\mathbb{K}^2$ different from $(0,0)$.

Answer: The system (2.1) has a non trivial solution $\iff$ $a_{00}a_{11} - a_{10}a_{01} = 0$. □

This is one of the first examples of what we call "elimination theory" in Commutative Algebra and Algebraic Geometry: one starts with two polynomials in 6 variables $a_{00}x_0+a_{01}x_1$, $a_{10}x_0+a_{11}x_1 \in \mathbb{K}[a_{00}, a_{01}, a_{10}, a_{11}, x_0, x_1]$ and by some formalism which seems on one side very familiar to any reader who has passed through a course of Linear Algebra but yet strange from the Computational Algebra point of view, we obtain another polynomial depending on less variables ($a_{00}a_{11} - a_{10}a_{01} \in \mathbb{K}[a_{00}, a_{01}, a_{10}, a_{11}]$) from which we can deduce some geometric properties of the system (2.1).

Problem 2.1 has several generalizations, some of them very likely to be familiar. Let us take a look at them.

Problem 2.2. (Increasing the number of variables) Find "the condition" for the following system of equations

$$\begin{cases} a_{00}x_0 + a_{01}x_1 + \ldots + a_{0n}x_n &= 0 \\ a_{10}x_0 + a_{11}x_1 + \ldots + a_{1n}x_n &= 0 \\ \vdots & \vdots \ \vdots \\ a_{n0}x_0 + a_{n1}x_1 + \ldots + a_{nn}x_n &= 0 \end{cases}$$

to have a solution in $\mathbb{K}^{n+1}$ different from $(0, 0, \ldots, 0)$.

Problem 2.3. (Increasing the degrees) Let d_0, $d_1 \in \mathbb{Z}_{\geq 1}$. Find "the condition" under which the following system of homogeneous polynomials

$$\begin{cases} a_{00}x_0^{d_0} + a_{01}x_0^{d_0-1}x_1 + \ldots + a_{0d_0}x_1^{d_0} &= 0 \\ a_{10}x_0^{d_1} + a_{11}x_0^{d_1-1}x_1 + \ldots + a_{1d_1}x_1^{d_1} &= 0 \end{cases}$$

has a solution in $\mathbb{K}^2 \setminus \{(0, 0)\}$.

Problem 2.4. (Increasing the degrees AND the number of variables) Let $n \in \mathbb{N}$, and $d_0, \ldots, d_n \in \mathbb{Z}_{\geq 1}$, find the condition under which the following system of homogeneous polynomials

$$\begin{cases} \sum_{\alpha_0+\ldots+\alpha_n=d_0} a_{0,\alpha_0,\ldots,\alpha_n} x_0^{\alpha_0} \ldots x_n^{\alpha_n} &= 0 \\ \sum_{\alpha_0+\ldots+\alpha_n=d_1} a_{1,\alpha_0,\ldots,\alpha_n} x_0^{\alpha_0} \ldots x_n^{\alpha_n} &= 0 \\ \vdots & \vdots \ \vdots \\ \sum_{\alpha_0+\ldots+\alpha_n=d_n} a_{n,\alpha_0,\ldots,\alpha_n} x_0^{\alpha_0} \ldots x_n^{\alpha_n} &= 0 \end{cases}$$

Homework 2.5.

(1) Prove carefully that the solution of Problem 2.2 is given by "the polynomial" $\det(a_{ij})$.

(2) Is it true that the solution of Problem 2.3 is given by the "Sylvester resultant"

$$\text{Resultant}_t(a_{00} + a_{01}t + \ldots + a_{0d_0}t^{d_0}, a_{10} + a_{11}t + \ldots + a_{1d_1}t^{d_1})?$$

Questions:

- Is there always "a condition" to solve all these problems?
- Is the condition independent of the ground field $\mathbb{K}$?
- Do we need to use always homogeneous polynomials?

One can get rid of the constrains of homogeneity of the polynomials and "same number of equations than unknowns" and get the more general situation:

Problem 2.6. Denote with $\boldsymbol{a}$ a list of parameters $\boldsymbol{a} = (a_1, \ldots, a_N)$. For $k,\, n \in \mathbb{N}$ let $f_k(\boldsymbol{a}, x_1, \ldots, x_n) \in \mathbb{K}[\boldsymbol{a}, x_1, \ldots, x_n]$. Find conditions on $\boldsymbol{a}$ such that the system

$$\begin{cases} f_1(\boldsymbol{a}, x_1, \ldots, x_n) & = & 0 \\ f_2(\boldsymbol{a}, x_1, \ldots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ f_k(\boldsymbol{a}, x_1, \ldots, x_n) & = & 0 \end{cases} \tag{2.2}$$

has a solution in $\mathbb{K}^n$.

Homework 2.7.

(3) Show that if the family (2.2) is a set of linear homogeneous equations of the form

$$\begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + \ldots + a_{2n}x_n & = & 0 \\ \vdots & \vdots & \vdots \\ a_{kn}x_1 + \ldots + a_{kn}x_n & = & 0 \end{cases}$$

with $k \geq n$, then there is a non trivial solution if and only if all the maximal minors of the matrix $\left(a_{ij}\right)_{1\leq i\leq k,\, 1\leq j\leq n}$ are zero.

Solving the "general" problem 2.6 can be a very complicated task, already in its simplest expressions. Consider for instance the case $k = n = 1$, so (2.2) boils down to

$$a_0 + a_1x_1 + a_2x_1^2 + \ldots + a_dx_1^d = 0. \tag{2.3}$$

Note that in this case, if $d > 1$, the answer to the question whether there is a root of this polynomial depends strongly on the field $\mathbb{K}$, and even if one restricts the attention to solutions in algebraically closed field (like the field of complex numbers $\mathbb{C}$), there are not "closed" conditions like "there will always be a solution" because one of the instances of (2.3) is $a_0 = 0$ (i.e. $a_1 = a_2 = \ldots = a_d = 0$), which does not have solutions if $a_0 \neq 0$. In this case, by homogenizing (2.3) we can get rid of this kind of odd situation, because the system

$$a_0x_0^d + a_1x_0^{d-1}x_1 + a_2x_0^{d-2}x_1^2 + \ldots + a_dx_1^d = 0$$

has always a complex nontrivial solution, independently of the values of $a_0, \ldots, a_d \in \mathbb{C}$.

To some extent, looking at homogeneous equations with coefficients in an algebraically closed field simplifies a lot our study on conditions over the system, and this is not a mere intuitive argument.

Let us look at the geometry of Problem 2.6. We have the following situation:

$$\begin{array}{ccc} V := \{(\boldsymbol{a}, x_1, \ldots, x_n) :\, f_1(\boldsymbol{a}, x_1, \ldots, x_n) = \ldots = f_k(\boldsymbol{a}, x_1, \ldots, x_n) = 0\} & \subset & \mathbb{K}^N \times \mathbb{K}^n \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{K}^N \end{array}$$

"The conditions" to solve (2.2) should be given by the equations of the image of the projection above or -from a more algebraic point of view- the polynomials defining the ideal of this variety. But the problem is that projections do not behave well when we work with algebraic sets like the one above, so $\pi_1(V)$ may not be described by polynomials! This is when Commutative Algebra and Algebraic Geometry come in our help, because if our variety above were a projective one (contained in $\mathbb{P}^n$ instead of $\mathbb{K}^n$), the we would get the following:

Theorem 2.8. [CLO97, Has07, Projective Elimination] *If $\mathbb{K}$ is algebraically closed and $V \subset \mathbb{K}^N \times \mathbb{P}^n$ is an algebraic set (i.e. a the solution of a system of polynomial equations which are homogeneous in the last $n+1$ variables), then $\pi_1(V) \subset \mathbb{K}^N$ is also an algebraic set.*

Question 2.9. Given $V \subset \mathbb{K}^N \times \mathbb{P}^n$ as before. Which are the equations of $\pi_1(V)$?.

Theorem 2.10. [CLO97, Has07, Algebraic version of the Projective Elimination] *If $\mathbb{K}$ is algebraically closed and $V \subset \mathbb{K}^N \times \mathbb{P}^n$ the variety given by the ideal $I = \langle f_1(\boldsymbol{a}, x_0, \ldots, x_n), \ldots, f_k(\boldsymbol{a}, x_0, \ldots, n)\rangle \subset \mathbb{K}[\boldsymbol{a}, x_0, \ldots, x_n]$ (these polynomials must be homogeneous in the variables $x_0, \ldots, x_n$) then $\pi_1(V)$ is the variety given by the zeroes of the ideal*

$$\widehat{I} := \{f \in \mathbb{K}[\boldsymbol{a}] : \textit{for each } 0 \leq i \leq n, \exists e_i \geq 0 \textit{ with } x_i^{e_i} f \in I\} \subset \mathbb{K}[\boldsymbol{a}]. \tag{2.4}$$

Example 2.11. Suppose the ideal I is given by $\langle f_0, f_1\rangle \subset \mathbb{K}[a_{00}, a_{01}, a_{10}, a_{11}, x_0, x_1]$ with $f_0 = a_{00}x_0 + a_{01}x_1$, $f_1 = a_{10}x_0 + a_{11}x_1$, and $\mathbb{K}$ algebraically closed. Let us see that the determinant $\Delta := a_{00}a_{11} - a_{10}a_{01}$ is an element of $\widehat{I}$. Indeed, we have that $a_{10}f_0 - a_{00}f_1 = -x_0\Delta$, and $a_{11}f_0 - a_{01}f_1 = x_1\Delta$, so $\Delta \in \widehat{I}$.

Of course it would be interesting to characterize completely the ideal $\widehat{I}$ in all cases, even in the one given by Example 2.11.

Question 2.12. Is it true that $\widehat{I} = \langle\Delta\rangle$ in this case? How do we compute $\widehat{I}$ if we have the generators of I.

Homework 2.13.

(4) Show that for $\pi_1 : \mathbb{K} \times \mathbb{K} \to \mathbb{K}$ given by $\pi_1(x_1, x_2) = x_1$, and $V(x_1x_2 - 1) \subset \mathbb{K}^2$, $\pi_1(V) \subset \mathbb{K}$ is not an algebraic set.

2.1. **A Session with `Mathematica`.** The ideal $\widehat{I}$ defined in (2.4) can be described algorithmically, and there are good formulas using elimination to describe it, see for instance [CLO97, Has07]. Let us see how a session with a Computational Algebra tool work if we use Gröbner bases to try to eliminate the homogeneous variables.

Example 2.14. We start with the polynomials of Example 2.11, and we try to "eliminate" the variables x_0 and x_1 with a suitable Lexicographic Monomial Order:

$f0 := a00 * x0 + a01 * x1$
$f1 = a10 * x0 + a11 * x1$
Factor[GroebnerBasis[$\{f0, f1\}, \{x0, x1, a00, a01, a10, a11\}$]]
$\{-(a01a10 - a00a11)x1, a10x0 + a11x1, a00x0 + a01x1\}$
We try a different monomial order
Factor[GroebnerBasis[$\{f0, f1\}, \{x1, x0, a00, a01, a10, a11\}$]]
$\{-(a01a10 - a00a11)x0, a10x0 + a11x1, a00x0 + a01x1\}$

Example 2.15. If we look for conditions for the following polynomials to have a nontrivial root in $\mathbb{C}^2$, we proceed as before:

$f0 := a00 * x0^2 + a01 * x0 * x1 + a02 * x1^2$
$f1 := a10 * x0^2 + a11 * x0 * x1 + a12 * x1^2$

Factor[GroebnerBasis[{f0, f1}, {x1, x0, a00, a01, a02, a10, a11, a12}]]

$$\left\{\begin{array}{l} \mathrm{x0}^3\left(\mathrm{a00}^2\mathrm{a12}^2-\mathrm{a00a01a11a12}-2\mathrm{a00a02a10a12}+\mathrm{a00a02a11}^2+\mathrm{a01}^2\mathrm{a10a12}-\mathrm{a01a02a10a11}+\mathrm{a02}^2\mathrm{a10}^2\right),\\ -\mathrm{x0}(-\mathrm{a00a12x0}-\mathrm{a01a12x1}+\mathrm{a02a10x0}+\mathrm{a02a11x1}),\\ -\mathrm{x0}^2(-\mathrm{a00a11x0}-\mathrm{a00a12x1}+\mathrm{a01a10x0}+\mathrm{a02a10x1}),\\ -\mathrm{x0}^2\left(\mathrm{a00}^2\mathrm{a12x0}-\mathrm{a00a01a11x0}-\mathrm{a00a02a10x0}-\mathrm{a00a02a11x1}+\mathrm{a01}^2\mathrm{a10x0}+\mathrm{a01a02a10x1}\right),\\ \mathrm{a10x0}^2+\mathrm{a11x0x1}+\mathrm{a12x1}^2,\\ \mathrm{a00x0}^2+\mathrm{a01x0x1}+\mathrm{a02x1}^2 \end{array}\right\}$$

We change the monomial order and obtain

Factor[GroebnerBasis[{f0, f1}, {x0, x1, a00, a01, a02, a10, a11, a12}]]

$$\left\{\begin{array}{l} \mathrm{x1}^3\left(\mathrm{a00}^2\mathrm{a12}^2-\mathrm{a00a01a11a12}-2\mathrm{a00a02a10a12}+\mathrm{a00a02a11}^2+\mathrm{a01}^2\mathrm{a10a12}-\mathrm{a01a02a10a11}+\mathrm{a02}^2\mathrm{a10}^2\right),\\ -\mathrm{x1}(-\mathrm{a00a12x0}-\mathrm{a01a12x1}+\mathrm{a02a10x0}+\mathrm{a02a11x1}),\\ -\mathrm{x1}^2(-\mathrm{a00a11x0}-\mathrm{a00a12x1}+\mathrm{a01a10x0}+\mathrm{a02a10x1}),\\ -\mathrm{x1}^2\left(\mathrm{a00}^2\mathrm{a12x0}-\mathrm{a00a01a11x0}-\mathrm{a00a02a10x0}-\mathrm{a00a02a11x1}+\mathrm{a01}^2\mathrm{a10x0}+\mathrm{a01a02a10x1}\right),\\ \mathrm{a10x0}^2+\mathrm{a11x0x1}+\mathrm{a12x1}^2,\\ \mathrm{a00x0}^2+\mathrm{a01x0x1}+\mathrm{a02x1}^2 \end{array}\right\}$$

Which essentially tell us that the polynomial

(2.5)
$$\mathrm{a00}^2\mathrm{a12}^2-\mathrm{a00a01a11a12}-2\mathrm{a00a02a10a12}+\mathrm{a00a02a11}^2+\mathrm{a01}^2\mathrm{a10a12}-\mathrm{a01a02a10a11}+\mathrm{a02}^2\mathrm{a10}^2$$

must be in the ideal $\widehat{I}$ in this case.

Homework 2.16.

(5) Prove that the polynomial (2.5) is the resultant of the polynomials $f0$ and $f1$ defined in Example 2.15.

(6) Carry out Gröbner bases computations similar to Examples 2.14 and 2.15 to find out equations for $\widehat{I}$ if I is any of the following:

- $I = \langle a_{00}x_0 + a_{01}x_1,\ a_{10}x_0 + a_{11}x_1,\ a_{20}x_0 + a_{21}x_1\rangle$
- $I = \langle a_{00}x_0 + a_{01}x_1,\ a_{10}x_0^3 + a_{11}x_0^2x_1 + a_{12}x_0x_1^2 + a_{13}x_1^3, \rangle$
- $I = \langle a_{00}x_0 + a_{01}x_1,\ a_{10}x_0^3 + a_{11}x_0^2x_1 + a_{12}x_0x_1^2 + a_{13}x_1^3,\ a_{20}x_0 + a_{21}x_1\rangle$

With a bit of effort, using the results given by `Mathematica` in both sessions 2.14 and 2.15, one can show that $\widehat{I}$ is actually generated by the polynomials Δ and (2.5) in each case. But of course we would like to be able to find "simpler" ways of computing these elements, like avoiding for instane the whole Gröbner basis calculation. In particular we would like to set us into a situation where

- $\widehat{I}$ is principal.
- $\widehat{I}$ is generated by an irreducible element.
- There is a "direct" way of computing this irreducible element.

In the next section we will see that the situation presented in Problem 2.4 (which contains Problems 2.1, 2.2 and 2.3) can be set within this context, and hence we have "a closed condition" to deal with all these cases. Of course one may think that this is a very limited situation, but as in the case of the determinant, the case of elimination in this kind of "ideal case" can also be adapted to more general cases, and also to find solutions of polynomial systems.

Homework 2.17.

(6) Prove that the polynomial (2.5) is an irreducible element in $\mathbb{C}[a00, a01, a02, a10, a11, a12]$.

3. Classical Elimination: homogeneous polynomials

We set ourselves in the situation of Problem 2.4. Let $n \in \mathbb{N}$, and $d_0, \ldots, d_n \in \mathbb{Z}_{\geq 1}$. For $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^n$, set $|\boldsymbol{\alpha}| = \sum_{j=0}^n \alpha_j$. Consider the "generic" homogeneous polynomials with respect to $d_0, \ldots, d_n$:

$$
\left\{ \begin{array}{lcl} f_0 & := & \sum_{|\boldsymbol{\alpha}|=d_0} a_{0,\boldsymbol{\alpha}} x_0^{\alpha_0} \ldots x_n^{\alpha_n} \\ f_1 & = & \sum_{|\boldsymbol{\alpha}|=d_1} a_{1,\boldsymbol{\alpha}} x_0^{\alpha_0} \ldots x_n^{\alpha_n} \\ \vdots & \vdots & \qquad\vdots \\ f_n & = & \sum_{|\boldsymbol{\alpha}|=d_n} a_{0,\boldsymbol{\alpha}} x_0^{\alpha_0} \ldots x_n^{\alpha_n} \end{array} \right. \tag{3.1}
$$

By "generic" we mean here that we will consider each of the $a_{i,\boldsymbol{\alpha}}$ as a new indeterminate over the field $\mathbb{C}$, i.e. $f_0, \ldots, f_n \in \mathbb{C}[a_{i,\boldsymbol{\alpha}}, x_0, \ldots, x_n, \, |\boldsymbol{\alpha}| = d_i, \, 0 \leq i \leq n]$. Note that they actually are elements of $\mathbb{Z}[a_{i,\boldsymbol{\alpha}}, x_0, \ldots, x_n, \, |\boldsymbol{\alpha}| = d_i, \, 0 \leq i \leq n]$. Denote with $N := \#\{a_{i,\boldsymbol{\alpha}}, \, |\boldsymbol{\alpha}| = d_i, \, 0 \leq i \leq n\}$.

The following results are classic and go all the way back to the times of Macaulay ([Mac02], although not formulated in this way), see also [Jou91, Jou97, CLO05].

Theorem 3.1. *The variety $V(f_0, \ldots, f_n) \subset \mathbb{C}^N \times \mathbb{P}^n$ is irreducible. It has dimension $N-1$.*

A nice fact is that the projection of irreducible varieties is irreducible, so we already have that our map $\pi_1 : \mathbb{C}^N \times \mathbb{P}^n \to \mathbb{C}^N$ is going to send the irreducible variety $V(f_0, \ldots, f_n)$ above to an irreducible variety in $\mathbb{C}^N$. But the situation is even better in this case:

Theorem 3.2. *The variety $\pi_1\big(V(f_0, \ldots, f_n)\big) \subset \mathbb{C}^N$ is irreducible, has codimension* 1, *it is defined over $\mathbb{Q}[a_{i,\boldsymbol{\alpha}}, x_0, \ldots, x_n, \, |\boldsymbol{\alpha}| = d_i, \, 0 \leq i \leq n]$ and, moreover, the map $\pi_1|_V : V \to \pi_1(V)$ is birational.*

Homework 3.3.

(7) Prove that number N defined above is $N = \binom{n+d_0}{n} + \binom{n+d_1}{n} + \ldots \binom{n+d_n}{n}$.

Definition 3.4. We call the (*homogeneous, dense, Macaulay*) *resultant* of the generic polynomials (3.1) to the irreducible element in $\mathbb{Z}[a_{i,\boldsymbol{\alpha}}, \, 0 \leq i \leq n]$ defining the ideal of $\pi_1\big(V(f_0, \ldots, f_n)\big)$. It is well-defined up to the sign. We denote it with $\mathrm{Res}_{d_0, \ldots, d_n}$.

3.1. Examples.

- The case of Problem 2.1 is actually $\mathrm{Res}_{1,1}$, which is equal to $\pm(a_{00}a_{11} - a_{10}a_{01})$.
- The situation presented in Problem 2.2 is now equivalent to studying the geometric elimination of $n+1$ homogeneous polynomials of degree 1, and hence we have that
$$\mathrm{Res}_{1,1,\ldots,1} = \pm \det\big(a_{ij}\big)_{0 \leq i,\, j \leq n}.$$
- In the case of Problem 2.3, we have that $n = 1$ and it is straightforward to check that Res_{d_0,d_1} = "The Sylvester Resultant" of f_0 and f_1.
- Let us compute a non-trivial example, we will do $\mathrm{Res}_{1,1,2}$ with `Mathematica`:
  f0:=a00x0 + a01x1 + a02x2
  f1:=a10x0 + a11x1 + a12x2
  f2:=a20x0^2 + a21x0x1 + a22x0x2 + a23x1^2 + a24x1x2 + a25x2^2

$MResultant[\{f0, f1, f2\}, \{x0, x1, x2\}]$

$$\begin{aligned}&\mathrm{a00^2a11^2a25 - a00^2a11a12a24 + a00^2a12^2a23 - 2a00a01a10a11a25 + a00a01a10a12a24}\\&\mathrm{+a00a01a11a12a22 - a00a01a12^2a21 + a00a02a10a11a24 - 2a00a02a10a12a23}\\&\mathrm{-a00a02a11^2a22 + a00a02a11a12a21 + a01^2a10^2a25 - a01^2a10a12a22 + a01^2a12^2a20}\\&\mathrm{-a01a02a10^2a24 + a01a02a10a11a22 + a01a02a10a12a21}\\&\mathrm{-2a01a02a11a12a20 + a02^2a10^2a23 - a02^2a10a11a21 + a02^2a11^2a20}\end{aligned}$$

3.2. **Properties.**

- Degrees: $\mathrm{Res}_{d_0,\ldots,d_n}$ is a homogeneous polynomial in the variables $a_{i,\boldsymbol{\alpha}}$, $0 \leq i \leq n$. Moreover, for a fixed i_0, $\mathrm{Res}_{d_0,\ldots,d_n}$ is a homogeneous polynomial in the variables $a_{i_0,\boldsymbol{\alpha}}$ of degree $\frac{d_0 \cdot d_1 \cdot \ldots d_n}{d_{i_0}}$.
- Specialization: For a given evaluation of the coefficients $a_{i,\boldsymbol{\alpha}} \mapsto \tilde{a}_{i,\boldsymbol{\alpha}}$ (which is equivalent to say given a specific choice of homogeneous polynomials $\tilde{f}_0, \ldots \tilde{f}_n \in \mathbb{C}[x_0, \ldots, x_n]$), we have $\mathrm{Res}_{d_0,\ldots,d_n}(\tilde{f}_0, \ldots, \tilde{f}_n) = 0$ if and only if there exists a zero of the system $\tilde{f}_0 = \tilde{f}_1 = \ldots = \tilde{f}_n = 0$ in $\mathbb{P}^n$.
- Poisson Formula: Denote with $f_1^0, \ldots, f_n^0$ the "part at infinity" of the polynomials $f_1, \ldots, f_n$ which is obtained by setting $x_0 \mapsto 0$ in these polynomials. Also, denote with $f_1^1, \ldots, f_n^1$ the "affinization" of the polynomials $f_1, \ldots, f_n$ by setting $x_0 \mapsto 1$. Then, we have that

$$\mathrm{Res}_{d_0,\ldots,d_n} = \mathrm{Res}_{d_1,\ldots,d_n}(f_1^0, \ldots, f_n^0)^{d_0} \prod_{\boldsymbol{\xi} \in V(f_1^1,\ldots,f_n^1)} f_0(\boldsymbol{\xi}).$$

- Additivity:

$$\mathrm{Res}_{d_0+d_0',d_1,\ldots,d_n}(f_0 \cdot f_0', f_1, \ldots, f_n) = \mathrm{Res}_{d_0,\ldots,d_n}(f_0, \ldots, f_n) \cdot \mathrm{Res}_{d_0',\ldots,d_n}(f_0', \ldots, f_n).$$

- Isobarism: If one "declares" that $\deg(a_{i,\alpha}) = \alpha \in \mathbb{N}^{n+1}$, then $\mathrm{Res}_{d_0,\ldots,d_n}$ is "homogeneous" of degree $d_0 \cdot d_1 \cdot \ldots \cdot d_n(1, 1, \ldots, 1)$.
- Extremal Coefficients: $\mathrm{Res}_{d_0,\ldots,d_n}$ has several well detected monomials with coefficients being ± 1.
- Hidden variables vs u-resultants If you have n homogeneous polynomials in n variables $f_1, \ldots, f_n \in \mathbb{K}[x_0, \ldots, x_n]$, for a given $i = 1, \ldots, n$ one can either compute
  - The "u"-resultant $\mathrm{Res}_{1,d_1,\ldots,d_n}(u_i x_0 - u_0 x_i, f_1, \ldots, f_n) \in \mathbb{K}[u_0, \ldots, u_n]$,
  - the polynomial $P_i(x_0, x_i) = \mathrm{Res}_{d_1,\ldots,d_n}(f_1^{(i)}, \ldots, f_n^{(i)})$, where $f_j^{(i)}$ is the polynomial consisting in giving "degree zero" to the variables x_0 x_i and rehomogenizing the polynomials again up to degree d_i. For instance, if $f = x_0^2 x_1^2 x_2 + x_1^5 + x_2^5 + x_0 x1 x2^3$, then $f^{(1)} = x_0 x_1^2 x_2 z^3 + x_1^5 z^5 + x_2^5 + x_0 x_1 x_2^3 z^2$, where z is the new homogenizating variable.

Proposition 3.5.

$$P(u_0, u_1) = \pm\, \mathrm{Res}_{1,d_1,\ldots,d_n}(u_i x_0 - u_0 x_i, f_1, \ldots, f_n),$$

and these two polynomials encode the i-th coordinates of the affine zeroes of the system $f_1^1, \ldots, f_n^1$.

The importance of properties like Proposition 3.5 can be highlighted with the following example.

Example 3.6. Consider the homogeneous polynomials $F_1(x_0,x_1,x_2)$, $F_2(x_0,x_1,x_2) \in \mathbb{C}[x_0,x_1,x_2]$, of respective degrees d_1 and d_2. The zero set of each of them gives a curve in projective space $\mathbb{P}^2$ and we would like to count properly the number of intersections. One can do of course $\mathrm{Res}_{d_1,d_2}(F_1^{(1)},F_2^{(2)})$, which would encode the 1-st coordinate of the roots, but then it is not clear from this operator applied to the sequence $F_1^{(1)},F_2^{(2)}$ which would be the degree of this polynomial in x_1. On the other hand, we know because of Proposition 3.5 that

$$\mathrm{Res}_{1,d_1,d_2}(u_0x_1-u_1x_0,F_1,F_2)$$

has degree d_1d_2 in the coefficients of (u_0,u_1), and moreover Poisson's formula tell us that up to a constant in $\mathbb{C}$, we have that this resultant is equal to

$$\prod_{\boldsymbol{\xi}\in V(F_1^{(1)}),F_2^{(2)})}(u_0\xi_1-u_1),$$

so we know that generically (if the constant we have not computed is not zero) the number of common intersections counted properly of these two curves in $\mathbb{P}^2$ is $d_1\cdot d_2$, so we recover Bézout's Theorem from the properties of the resultant.

Homework 3.7.

(8) Apply Poisson's formula to a system with $n=2$, $d_0=d_1=1$, $d_2=2$.

Open Problem: We know that there are several coefficients in the monomial expansion of $\mathrm{Res}_{d_0,\ldots,d_n}$ which are equal to ±1, but we **do not know** which is the largest nonzero integer appearing in this expression, i.e. we do not know its *arithmetic height*. This problem is of importance in Arithmetic Geometry and Dynamical systems (see for instance [DGS14]). Some bounds are given in [Som04, DKS13], but we do not know if they are sharp or not. Good conjectures are needed here!

3.3. **Formulas for computing** $\mathrm{Res}_{d_0,\ldots,d_n}$. In the case $n=1$, it should be familiar to the reader that Res_{d_0,d_1} can be computed as the determinant of a "Sylvester" type matrix of size (d_0+d_1). Indeed, if we set $\mathbb{K}=\mathbb{Q}(a_{i,\boldsymbol{\alpha}},\,|\boldsymbol{\alpha}|=d_i,\,i=0,1)$, then we have the following $\mathbb{K}$-linear map:

$$\begin{array}{rccc}\varphi: & \mathbb{K}[x_0,x_1]_{d_1-1}\oplus\mathbb{K}[x_0,x_1]_{d_0-1} & \to & \mathbb{K}[x_0,x_1]_{d_0+d_1-1}\\ & (g_0,g_1) & \mapsto & g_0f_0+g_1f_1.\end{array}\tag{3.2}$$

Note that φ is a map between $\mathbb{K}$-vector spaces of the same dimension, and that its matrix in the standard monomial bases is the Sylvester matrix.

Theorem 3.8.

$$\det(\varphi)=\pm\,\mathrm{Res}_{d_0,d_1}$$

From here, we can see easily that $\det(\varphi)=0$ for a particular specialization of the coefficients if and only if $\tilde{f}_0$ and $\tilde{f}_1$ have a common factor of positive degree, which is equivalent over $\mathbb{C}$ to have a common zero in $\mathbb{P}^1$. This condition of having a common factor if and only if having a common zero is only proper of the case $n=1$.

If $n>1$, there is a general version of the map (3.2) which we describe as follows: set $\mathbb{K}=\mathbb{Q}(a_{i,\boldsymbol{\alpha}},\,|\boldsymbol{\alpha}|=d_i,\,0\le i\le n)$, $D=\left(\sum_{i=0}^n d_i\right)-n$, and consider the following map of $\mathbb{K}$-vector spaces

$$\begin{array}{rccc}\Phi: & \oplus_{i=0}^n\mathbb{K}[x_0,\ldots,x_n]_{D-d_i} & \to & \mathbb{K}[x_0,\ldots,x_n]_D\\ & (g_0,\ldots,g_n) & \mapsto & g_0f_0+\ldots+g_nf_n.\end{array}\tag{3.3}$$

Theorem 3.9. *The map Φ is surjective, and the* gcd *of the maximal minors of its matrix in the monomial bases is equal to* $\pm\,\mathrm{Res}_{d_0,\ldots,d_n}$.

If one wants a more "clear" formulation for $\mathrm{Res}_{d_0,\ldots,d_n}$ as a "determinant", then one can see the formulation given by Cayley and highlighted in Appendix A of [GKZ94] of the resultant as the determinant of a "complex" of vector spaces. In that sense, the map (3.3) can be regarded as the last part of a longer complex of exact $\mathbb{K}$-vector spaces whose "determinant" will be equal to $\mathrm{Res}_{d_0,\ldots,d_n}$ up to the sign.

Homework 3.10.
(9) Compute the matrix in the monomial bases of the map Φ of (3.3) in the following cases: $d_0 = d_1 = 1$, $d_2 = 2(n = 2)$ and $d_0 = d_1 = d_2 = 2$.

Macaulay gave somehow an intriguing contribution to making explicit the computation of $\mathrm{Res}_{d_0,\ldots,d_n}$ as the quotient of a specific (well-defined) minor of (3.3) divided by another minor of the same matrix. The original work can be found in [Mac02], see also [DD01, CLO05].

There are also other matricial formulations than those coming from (3.3) for computing $\mathrm{Res}_{d_0,\ldots,d_n}$ involving matrices "a la Bézout", Dixon-like, hybrid types and other methods, you can find several of them in [Jou97, DD01, EM99, EM07].

3.4. **Software.** None of the standard current Symbolic Computation Software has a command for computing $\mathrm{Res}_{d_0,\ldots,d_n}$ directly, most of them have a `Resultant` sentence which computes the Sylvester Resultant of two affine polynomials in one variable or -in our language- two homogeneous polynomials in two variables. The following is a (not exhaustive) list of codes available for producing resultant matrices and having some kind of "representation" of the resultant. Note that the number of coefficients in the monomial expansion of the resultant can be a huge number, so its calculation is a heavy task even in the more elementary cases.

- Several codes in `C, Maple` and `Matlab` for manipulating polynomials in several variables and computing resultant matrices, available at
  `http://www-sop.inria.fr/galaad/logiciels/emiris/soft_alg.html`
  Author: Ioannis Emiris
- Multires – a `Maple` package for the manipulation of multivariate polynomials, containing several tools for resultants, residues and the resolution of polynomial systems. Available at `https://www-sop.inria.fr/teams/galaad/software/multires/`
  Authors: Laurent Busé and Bernard Mourrain.
- EliminationMatrices – A package for computing resultants in `Macaulay2,` available at `http://www.math.uiuc.edu/Macaulay2/doc/Macaulay2-1.7/share/doc/Macaulay2/EliminationMatrices/html/`
  Authors: Nicolás Botbol, Laurent Busé, and Manuel Dubinski
- Java package for computing modular determinants and constructing Macaulay matrices, available at `http://works.bepress.com/minimair/30/`
  Authors: Manfred Minimair, Sarah Smith, Jonathan Curran, and Julio Macavilca.

Homework 3.11.
(10) Try any of the codes above to compute $\mathrm{Res}_{2,2,2}$.

4. Sparse Elimination Theory

In practice one does not start with full homogeneous polynomials $f_0, \ldots, f_n$ as in (3.1), but with functions with more "structure" like in the following example.

Example 4.1. If $n > 1$, and the affine polynomials $f_0^1, \ldots, f_n^1 \in \mathbb{C}[x_1, \ldots, x_n]$ are such that their degree in each individual variable x_i, $1 \leq i \leq n$ is bounded by d_i, i.e. we have a situation like the following:

$$f_i^1 = \sum_{\alpha_j \leq d_i,\, 1 \leq j \leq n} a_{i,\boldsymbol{\alpha}} x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ 0 \leq i \leq n \tag{4.1}$$

then by homogenizing them and applying $\mathrm{Res}_{d,\ldots,d}$ to this family, where $d = d_1 + \ldots + d_n$, we will find that the resultant will vanish identically, and this is be due to the fact that the point $(0 : 1 : 0 : \ldots : 0) \in \mathbb{P}^n$ is a common zero of the homogenized system (4.1).

Homework 4.2.

(11) Show that $(0 : 1 : 0 : \ldots : 0) \in \mathbb{P}^n$ is a common zero of the homogenizations of (4.1). Can you find more common zeroes?

A way of working around this situation is by considering "multihomogeneous" polynomials, i.e. changing the homogenization: we would pass from a situation like (4.1) to the following system of multihomogeneous polynomials:

$$f_i = \sum_{\alpha_j \leq d_i,\, 1 \leq j \leq n} a_{i,\boldsymbol{\alpha}} x_1^{\alpha_1} y_1^{d_i - \alpha_1} \cdots x_n^{\alpha_n} y_n^{d_i - \alpha_n} \in \mathbb{Q}[a_{i,\boldsymbol{\alpha}}, x_1, \ldots, x_n, y_1, \ldots, y_n],\ 0 \leq i \leq n, \tag{4.2}$$

These functions are multihomogeneous in the sense that they are homogeneous in each group of variables (x_i, y_i), $1 \leq i \leq n$, and one could consider their set of zeroes in the multiprojective space $\mathbb{P}^1 \times \mathbb{P}^1 \times \ldots \times \mathbb{P}^1$ instead of $\mathbb{P}^n$ as we did before. This leads to a very meaningful theory as elimination theory works very well also if we replace the projective space with a multiprojective one. But our polynomials can be even more "sparse" than those in (4.1). What happens for instance with a system like

$$\left\{ \begin{array}{lcl} f_0^1 & = & a_{01} + a_{02} x_1^2 x_2^2 + a_{03} x_1 x_2^3 \\ f_1^1 & = & a_{10} + a_{11} x_1^2 + a_{12} x_1 x_2^2 \\ f_2^1 & = & a_{20} x_1^3 + a_{21} x_1 x_2. \end{array} \right. \quad ? \tag{4.3}$$

Homework 4.3.

(12) Show that by either homogeneizing this system to consider its zeroes either in $\mathbb{P}^2$ or in $\mathbb{P}^1 \times \mathbb{P}^1$, we get nontrivial solutions of them (i.e. both the homogeneous and the bi-homogeneous resultant will vanish).

Is there an "homogenization" for the system given in (4.3) which can lead us to a meaningful theory of resultants there? Sparse elimination theory deals with situations like the above. Instead of pre-declaring the total degrees or some partial degrees of the input polynomials, we will fix their *support*, i.e. the non-zero exponents appearing in the monomial expansion of each f_i.

To get started in this direction, consider $n + 1$ finite subsets of $\mathbb{Z}^n$ which will be our set of exponents: $\mathcal{A}_0, \ldots, \mathcal{A}_n \subset \mathbb{Z}^n$. The reason we now allow negative exponents will be clarified soon. For $i = 0, \ldots, n$, consider the generic polynomial with support in $\mathcal{A}_i$:

$$f_i = \sum_{\boldsymbol{\alpha}\in\mathcal{A}_i} a_{i,\boldsymbol{\alpha}} x_1^{\alpha_1}\dots x_n^{\alpha_n} \in \mathbb{Z}[a_{j\boldsymbol{\alpha}}, x_1^{\pm1},\dots,x_n^{\pm1}, \boldsymbol{\alpha}\in\mathcal{A}_j,\, 0\le j\le n]. \tag{4.4}$$

In the case of Example 4.3 we have $\mathcal{A}_0=\{(0,0),\,(2,2),\,(1,3)\}$, $\mathcal{A}_1=\{(0,0),\,(2,0),\,(1,2)\}$, and $\mathcal{A}_2=\{(3,0),\,(1,1)\}$. In the previous section (homogeneous polynomials), we have that our input polynomials can gotten by setting $x_0\mapsto 1$ in each of the f_i, $i=0,\dots,n$, and the finite sets $\mathcal{A}_i$ are the following:

$$\mathcal{A}_i=\{\boldsymbol{\alpha}\in\mathbb{N}^n:\,|\boldsymbol{\alpha}|\le d_i\},\ 0\le i\le n.$$

Homework 4.4.

(13) Describe the sets $\mathcal{A}_i$, $0\le i\le n$ for the family $f_0^1,\dots,f_n^1$ defined in (4.1).

Before going through the "homogenization" which will require some formalism, let us take a look at the geometry. Because we allow now negative exponents, the set of zeroes must be taken in the n-dimensional *torus* $(\mathbb{C}^\times)^n$, where $\mathbb{C}^\times:=\mathbb{C}\setminus\{0\}$. Set $N=\sum_{j=0}^n \#\mathcal{A}_i$, and denote with $\boldsymbol{x}=(x_1,\dots,x_n)$ for short. As before, we have the following diagram:

$$\begin{array}{ccc} V=\{(a_{i,\boldsymbol{\alpha}},\boldsymbol{x}):\, f_0(a_{0,\boldsymbol{\alpha}},\boldsymbol{x})=\cdots=f_n(a_{n,\boldsymbol{\alpha}},\boldsymbol{x})=0\} & \subset & \mathbb{C}^N\times(\mathbb{C}^\times)^n \\ \downarrow\pi_1 & & \downarrow\pi_1 \\ \pi_1(V) & \subset & \mathbb{C}^N. \end{array} \tag{4.5}$$

We cannot use the Projective closure Theorem 2.8 here as $(\mathbb{C}^\times)^n$ is not the projective space, or any kind of compact projective variety where this result will hold. We will see soon that there is a suitable compactification of the torus which takes very well into account the information codified by the data $(\mathcal{A}_0,\dots,\mathcal{A}_n)$, and generalizes the projective space in the homogeneous case. But it is not hard to prove the following:

Proposition 4.5. *The ideal* $\langle f_0,\dots,f_n\rangle\subset\mathbb{C}[a_{i,\boldsymbol{\alpha}},x_1^{\pm1},\dots,x_n^{\pm1},\,\boldsymbol{\alpha}\in\mathcal{A}_i,\,0\le i\le n]$ *is prime and of dimension* $N-1$.

This claim gives us some hope to find some irreducible equation defining either $\pi_1(V)$ or its Zariski closure, as it is very reasonable to hope that $\pi_1(V)$ (or its algebraically closure) will be an irreducible variety of codimension one in $\mathbb{C}^N$, and that would be the resultant that we are looking for.

Unfortunately, the situation is not very straightforward here. Let us see some of the non trivial situations that we have to navigate in order to get something "reasonably" defined as our sparse resultant.

Example 4.6.

- Set $n=2$, $\mathcal{A}_0=\mathcal{A}_1=\mathcal{A}_2=\{(0,0),\,(1,1)\}$, and write
$$f_i=a_{i,00}+a_{i11}x_1x_2,\ i=0,1,2.$$
It is straightforward to see that the following polynomials vanish in $\pi_1(V)$: $a_{0,00}a_{1,11}-a_{0,11}a_{1,00}$, and $a_{1,00}a_{2,11}-a_{1,11}a_{2,00}$. This shows that the variety $\pi_1(V)\subset\mathbb{K}^6$ does not have codimension 1 in $\mathbb{K}^6$, and that there will not be "a" resultant in this case.
- Suppose $n=2$ and set $\mathcal{A}_0=\mathcal{A}_1=\{(0,0),\,(1,1)\}$, $\mathcal{A}_2=\{(0,0),\,(1,0),\,(0,1)\}$. As above, we have that the element $a_{0,00}a_{1,11}-a_{0,11}a_{1,00}$ belongs to the ideal of $\pi_1(V)$, and surprisingly it is the irreducible generator of this ideal. Note that the "resultant" in this case does not depend on the coefficients of f_2.

- Set again $n = 2$ and $\mathcal{A}_0 = \mathcal{A}_1 = \mathcal{A}_2 = \{(0,0),\ (2,0),\ (0,2)\}$, and write

$$f_i = a_{i,00} + a_{i,20}x_1^2 + a_{i,02}x_2^2, \quad i = 0,1,2. \tag{4.6}$$

In this case it is not hard to check that $\pi_1(V)$ is defined by the irreducible polynomial

$$\det\begin{pmatrix} a_{0,00} & a_{0,20} & a_{0,02} \\ a_{1,00} & a_{1,20} & a_{1,02} \\ a_{2,00} & a_{2,20} & a_{2,02} \end{pmatrix},$$

but the map $\pi_1|_V$ has degree 4. This observation may seem irrelevant, but we will see that it will play a central rol in the definition of the resultant.

Whether $\pi_1(V)$ is an algebraic set or not, we can consider its Zariski closure $\overline{\pi_1(V)} \subset \mathbb{K}^N$, which we know it is irreducible thanks to Proposition 4.5, and asks whether it is of codimension one or not. There is a very nice combinatorial criterion to decide this: for $i = 0, \ldots, n$, set

$$\mathcal{A}_i = \{a_{i,0}, \ldots, a_{i,m_i}\} \subset \mathbb{Z}^n, \tag{4.7}$$

and consider the sublattice given by

$$L_{\mathcal{A}_i} = \sum_{j=1}^{c_i}(a_{i,j} - a_{i,0})\mathbb{Z}.$$

Clearly, this lattice does not depend on the choice of $a_{i,0} \in \mathbb{Z}$. Set now $L_{\boldsymbol{\mathcal{A}}} = \sum_{i=0}^m L_{\mathcal{A}_i}$. With more generality, for any $J \subset \{0, \ldots, n\}$, we set

$$L_{\boldsymbol{\mathcal{A}}_J} = \sum_{j\in J} L_{\mathcal{A}_j}$$

The following combinatorial criterion was found by Sturmfels in [Stu94] (see also [DS15]):

Proposition 4.7. *The variety $\overline{\pi_1(V)}$ has codimension 1 in $\mathbb{C}^N$ if and only if*

$$\mathit{rank}(L_{\boldsymbol{\mathcal{A}}_J}) \geq \#J - 1 \text{ for all } J \subset \{0, \ldots, n\}.$$

This statement is computationally "heavy" to check but give us at least a basic linear algebra criterion over vectors of $\mathbb{Z}^n$ to decide if we will have a resultant or not.

Homework 4.8.

(14) Use the criteria given by Proposition 4.7 to decide in which cases $\overline{\pi_1(V)}$ has codimension 1 in all the cases of Example 4.6.
(15) Do the same with the supports of the system given in Example 4.3.
(16) If

$$\mathcal{A}_0 = \{(0,0,0,0)\},\ \mathcal{A}_i = \{(0,0,0,0),\ (1,0,0,0),\ (0,1,0,0),\ (0,0,1,0),\ (0,0,0,1)\},\ i = 1,2,3,4, \tag{4.8}$$

can you say if $\overline{\pi_1(V)}$ has codimension 1 or not? If the answer is "yes", can you find the irreducible element generating this variety?

There is a bit more we can say from a combinatorial point of view. Set $\boldsymbol{\mathcal{A}} = \{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ for short.

Definition 4.9. Let $J \subset \{0, \ldots, n\}$. The subfamily $\boldsymbol{\mathcal{A}}_J = (\mathcal{A}_j)_{j\in J}$ is *essential* if the following conditions hold:

(1) $\#J = \operatorname{rank}\left(L_{\boldsymbol{\mathcal{A}}_J}\right) + 1$;
(2) $\#J' \leq \operatorname{rank}\left(L_{\boldsymbol{\mathcal{A}}'_J}\right)$ for all $J' \subsetneq J$.

Theorem 4.10 ([Stu94, DS15]). *The codimension of $\overline{\pi_1(V)}$ is equal to 1 if and only if there exists a unique essential subfamily $\boldsymbol{\mathcal{A}}_J$ of $\boldsymbol{\mathcal{A}}$. If this is the case, the polynomial defining this variety depends only on the coefficients of the polynomial f_j, $j \in J$.*

Homework 4.11.

(17) Check for the unique essential family (in the cases where there is only one) in (4.8) and Examples 4.3 and 4.6.

5. Homogenizing the equations: toric varieties

Here is where we will finally replace the torus $(\mathbb{C}^\times)^n$ in (4.5) with a suitable (projective, compact) variety $X_{\boldsymbol{\mathcal{A}}}$ from which we will be able to use a general version of Projective Elimination Theory to conclude that $\pi_1(V)$ is now a variety, and interpret the vanishing of the "resultant" as a condition to have common zeroes in $X_{\boldsymbol{\mathcal{A}}}$, i.e. we will see how to "homogenize" the f_i from (4.4) into some $\tilde{f}_i$ in order to have

$$\begin{array}{ccc} V = \{(a_{i,\boldsymbol{\alpha}}, \boldsymbol{X}) :\ \tilde{f}_0(a_{0,\boldsymbol{\alpha}}, \boldsymbol{X}) = \cdots = \tilde{f}_n(a_{n,\boldsymbol{\alpha}}, \boldsymbol{X}) = 0\} & \subset & \mathbb{C}^N \times X_{\boldsymbol{\mathcal{A}}} \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{C}^N. \end{array}$$

This is important as we would like a criteria as the following: *the resultant vanishes if and only if the $\tilde{f}_i$ have a common zero in ???*, and for this we need to know exactly what it means to be in $\pi_1(V)$.

$X_{\boldsymbol{\mathcal{A}}}$ will be a *toric variety* in the sense of [Ful93] (see also [CLS11]), which we will describe explicitely: recall from (4.7) that m_i is the cardinality of the set $\mathcal{A}_i$, and consider the following monomial map:

$$\begin{array}{rcl} \varphi_{\mathcal{A}} \colon (\mathbb{C}^\times)^n & \to & \mathbb{P}^{m_0} \times \ldots \mathbb{P}^{m_n} \\ \xi & \mapsto & \left(\left(\xi^{a_{0,0}} : \ldots : \xi^{a_{0,m_0}}\right), \ldots, \left(\xi^{a_{n,0}} : \ldots : \xi^{a_{n,m_n}}\right)\right). \end{array}$$

We set our variety $X_{\mathcal{A}}$ to be $\overline{\varphi_{\mathcal{A}}\left((\mathbb{C}^\times)^n\right)}$. It is a multiprojective toric subvariety in the sense of [CLS11], and we know that

$$\dim(X_{\boldsymbol{\mathcal{A}}}) = \operatorname{rank}(L_{\boldsymbol{\mathcal{A}}}).$$

With this in mind, consider the following diagram

(5.1)
$$\begin{array}{ccc} V = \{(a_{i,\boldsymbol{\alpha}}, \boldsymbol{X}) :\ \tilde{f}_0(a_{0,\boldsymbol{\alpha}}, \boldsymbol{X}) = \cdots = \tilde{f}_n(a_{n,\boldsymbol{\alpha}}, \boldsymbol{X}) = 0\} & \subset & \mathbb{C}^N \times \mathbb{P}^{m_0} \times \ldots \mathbb{P}^{m_n} \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ \pi_1(V) & \subset & \mathbb{C}^N \end{array},$$

and apply Projective Elimination to this situation.

6. Sparse Eliminants and Resultants

Now we are ready to define our objects. Recall that we know that $\pi_1(V)$ in (5.1) is an irreducible variety

Definition 6.1. The $\boldsymbol{\mathcal{A}}$-*eliminant* or *sparse eliminant*, denoted by $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$, is defined as any irreducible polynomial in $\mathbb{Z}[a_{i,\boldsymbol{\alpha}}]$ giving an equation for $\pi_1(V)$, if this is a hypersurface, and as 1 otherwise.

The $\boldsymbol{\mathcal{A}}$-*resultant* or *sparse resultant*, denoted by $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$, is defined as any primitive polynomial in $\mathbb{Z}[a_{i,\boldsymbol{\alpha}}]$ giving an equation for the direct image $\pi_1{*}V$.

Both the sparse eliminant and the sparse resultant are well-defined up to a sign. It follows from these definitions that there exists $d_{\boldsymbol{\mathcal{A}}} \in \mathbb{N}$ such that

$$\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \pm\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^{d_{\boldsymbol{\mathcal{A}}}}, \tag{6.1}$$

with $d_{\boldsymbol{\mathcal{A}}}$ equal to the degree of the restriction of π_1 to the incidence variety V.

Remark 6.2. The $\boldsymbol{\mathcal{A}}$-resultant is usually defined as an irreducible polynomial giving an equation for the Zariski closure $\pi_1(V)$, if this is a hypersurface, and equal to the constant 1 otherwise (see [GKZ94, Stu94]). This irreducible element is what we call here the $\boldsymbol{\mathcal{A}}$-eliminant.

The definition of the sparse resultant in terms of a direct image rather than just a set-theoretical image has better properties and produces more uniform statements as we will see soon. This is why we distinguish this concept by calling it *the geometric sparse resultant*, which will be denoted by "resultant" and denoted with $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ for the rest of the text.

Example 6.3. For the family (4.3), we have that the exponent $d_{\boldsymbol{\mathcal{A}}} = 1$ and hence both polynomials $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ and $\pm\,\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$ coincide. An explicit calculation of this polynomial gives:

(6.2)

$$\begin{aligned}
&\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = a_{01}^5a_{12}^7a_{20}^6a_{21} + 3a_{01}^4a_{02}a_{11}^2a_{12}^5a_{20}^4a_{21}^3 + 3a_{01}^3a_{02}^2a_{11}^4a_{12}^3\gamma_z^2a_{21}^5\\
&-13a_{01}^3a_{02}a_{03}a_{10}^2a_{11}a_{12}^4a_{20}^5a_{21}^2 - 7a_{01}^3a_{03}^2a_{10}a_{11}^3a_{12}^3a_{20}^4a_{21}^3 + 6a_{01}^2a_{02}^3a_{10}^3a_{11}a_{12}^3a_{20}^4a_{21}^3\\
&+a_{01}^2a_{02}^3a_{11}^6a_{12}a_{21}^7 - a_{01}^2a_{02}^2a_{03}a_{10}^2a_{11}^3a_{20}^3a_{21}^4 + 5a_{01}^2a_{02}a_{03}^2a_{10}^4a_{12}^3a_{20}^6a_{21}\\
&-a_{01}^2a_{02}a_{03}^2a_{10}a_{11}^5a_{12}a_{20}^2a_{21}^5 + 14a_{01}^2a_{03}^3a_{10}^3a_{11}^2a_{12}^2a_{20}^5a_{21}^2 + a_{01}^2a_{03}^3a_{11}^7a_{20}a_{21}^6\\
&-2a_{01}a_{02}^4a_{10}^3a_{11}^3a_{12}a_{20}^2a_{21}^5 - 5a_{01}a_{02}^3a_{03}a_{10}^5a_{12}^2a_{20}^5a_{21}^2 + a_{02}^5a_{10}^6a_{12}a_{20}^4a_{21}^3\\
&+2a_{01}a_{02}^2a_{03}^2a_{10}^4a_{11}^2a_{12}a_{20}^4a_{21}^3 - 2a_{01}a_{02}a_{03}^3a_{10}^3a_{11}^4a_{20}^3a_{21}^4 - 7a_{01}a_{03}^4a_{10}^5a_{11}a_{12}a_{20}^6a_{21}\\
&+a_{02}^2a_{03}^3a_{10}^6a_{11}a_{20}^5a_{21}^2 + a_{03}^5a_{10}^7a_{20}^7
\end{aligned}$$

Example 6.4. Consider the polynomials f_0, f_1, f_2 defined in (4.6). Then we have that

$$\mathrm{Elim}_{\boldsymbol{\mathcal{A}}} = \det\begin{pmatrix} a_{0,00} & a_{0,20} & a_{0,02}\\ a_{1,00} & a_{1,20} & a_{1,02}\\ a_{2,00} & a_{2,20} & a_{2,02}\end{pmatrix},$$

but it turns out that the exponent $d_{\boldsymbol{\mathcal{A}}}$ is equal to 4. This is due to the fact that for every nontrivial zero (t_1, t_2) of the system $f_0 = f_1 = f_2 = 0$, we actually have the following three (generically different) points also "living" in the variety V with the same values of $\boldsymbol{a}$: $(-t_1, t_2)$, $(t_1, -t_2)$, $(-t_1, -t_2)$. So, in this case, $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} = \mathrm{Elim}_{\boldsymbol{\mathcal{A}}}^4$.

Homework 6.5.

(18) Compute the eliminants of the two examples above (one idea could be to use a Gröbner basis over the family f_0, f_1, f_2 eliminating first the variables x_1 and x_2.

(19) Show that the exponent $d_{\boldsymbol{\mathcal{A}}}$ in Example 6.4 is equal to 4.

To compute explicitly the exponent $d_{\boldsymbol{\mathcal{A}}}$ in all the cases, we need to introduce some numerical invariantes which are going to replace the numbers $d_0, \ldots, d_n$ of the homogeneous case.

The *mixed volume* of a family of n compact bodies $Q_1, \ldots, Q_n \subset \mathbb{R}^n$ is defined as

$$\mathrm{MV}_n(Q_1, \ldots, Q_n) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1 \le i_1 < \cdots < i_j \le n} \mathrm{vol}_n(Q_{i_1} + \cdots + Q_{i_j}), \tag{6.3}$$

where $\mathrm{vol}_n(\cdot)$ is the standard volume in $\mathbb{R}^n$. It is not clear from (6.3) that $\mathrm{MV}_n(Q_1, \ldots, Q_n)$ is a positive number, but indeed it is actually a nonegative integer, and there are very efficient ways of computing it without having to go through this alternating sum.

Mixed volumes are of importance in intersection theory of toric varieties due to the celebrated Bernstein sharp bound on the number of roots of a system of sparse polynomials in the torus given in [Ber75], so it should be not surprising that they appear in this context. For $i = 0, \ldots, n$, set $\Delta_i \subset \mathbb{R}^n$ to be the convex hull of $\mathcal{A}_i$.

Proposition 6.6.

$$\deg_{a_{i,\boldsymbol{\alpha}}}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) = \mathrm{MV}_n(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n).$$

Example 6.7. For the system given in (4.3), a straightforward computation shows that

$$\mathrm{MV}_2(\Delta_0, \Delta_1) = 7, \mathrm{MV}_2(\Delta_0, \Delta_2) = 7,\ \mathrm{MV}_2(\Delta_1, \Delta_2) = 5,$$

which coincides with the homogeneities of the polynomial given in (6.2).

Example 6.8. For the polynomials defined in (4.6), we have that $\mathrm{MV}_2(\Delta_i, \Delta_j) = 4$ for all $i, j \in \{0, 1, 2\}$. This coincides with the partial degrees of the resultant in this case (but **not** of the eliminant!).

Now we can give an explicit description of the exponent $d_{\boldsymbol{\mathcal{A}}}$ which appears in (6.1). This formula already appears in [Est07, Theorem 2.23].

Proposition 6.9. *Suppose that* $\mathrm{Res}_{\boldsymbol{\mathcal{A}}} \neq 1$ *and let* $\boldsymbol{\mathcal{A}}_J$ *be the unique essential subfamily of* $\boldsymbol{\mathcal{A}}$. *Then*

$$d_{\boldsymbol{\mathcal{A}}} = [L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}} : L_{\boldsymbol{\mathcal{A}}_J}] \cdot \mathrm{MV}_{Z^n/L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}}}(\{\varpi(\Delta_i)\}_{i \notin J}),$$

where $L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}}$ *denotes the saturation of the sublattice* $L_{\boldsymbol{\mathcal{A}}_J}$ *in* $\mathbb{Z}^n$, *and* ϖ *the projection* $M_{\mathbb{R}} \to \mathbb{Z}^n / L_{\boldsymbol{\mathcal{A}}_J}^{\mathrm{sat}} \otimes \mathbb{R}$.

Homework 6.10.

(17) Compute the exponent $d_{\boldsymbol{\mathcal{A}}}$ in the Examples 4.3 and 4.6.

6.1. **Poisson formula.** There is also an analogue of the Poisson formula for these geometric sparse resultants, but their statement is a bit less straightforward than the one given in the classical case. To state it properly, we introduce some notation: let $v \in \mathbb{Z}^n \setminus \{0\}$ and put $v^\perp \simeq \mathbb{Z}^{n-1}$ for its orthogonal lattice. For $i = 1, \ldots, n$, we set $\mathcal{A}_{i,v}$ for the subset of points of $\mathcal{A}_i$ of minimal weight in the direction of v. This gives a family of n nonempty finite subsets of translates of the lattice $v^\perp$. We denote by $\mathrm{Res}_{\mathcal{A}_{1,v},\ldots,\mathcal{A}_{n,v}}$ the corresponding sparse resultant, also called the sparse resultant of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ in the direction of v. Set also $h_{\mathcal{A}_0}(v) = \min_{a\in\mathcal{A}_0}\langle v, a\rangle$ for the minimum value at v of the support function of $\mathcal{A}_0$.

Theorem 6.11. *(*[DS15]*) Let $\mathcal{A}_i \subset \mathbb{Z}^n$ be a nonempty finite subset, $i = 0, \ldots, n$. Then*

$$\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\ldots,\mathcal{A}_n} = \pm \prod_v \mathrm{Res}_{\mathcal{A}_{1,v},\ldots,\mathcal{A}_{n,v}}^{-h_{\mathcal{A}_0}(v)} \cdot \prod_\xi f_0(\xi),$$

the first product being over the primitive vectors $v \in \mathbb{Z}^n$ and the second over the roots ξ of $f_1, \ldots, f_n$ in the algebraic closure of $\mathbb{C}(a_{i,\boldsymbol{\alpha}},\, \boldsymbol{\alpha} \in \mathcal{A}_i,\, 1 \le i \le n)$.

Both products in the above formula are finite. Indeed, $\mathrm{Res}_{\mathcal{A}_{1,v},\ldots,\mathcal{A}_{n,v}} \neq 1$ only if v is an inner normal to a facet of the Minkowski sum $\sum_{i=1}^n \Delta_i$. Moreover, by Bernstein theorem [Ber75, Theorem B], the hypothesis that no directional sparse resultant vanishes implies that the set of roots of the family f_i, $i = 1, \ldots, n$, is finite.

Example 6.12. Let $M = \mathbb{Z}^2$ and consider the family of nonempty finite subsets of $\mathbb{Z}^2$

$$\mathcal{A}_0 = \mathcal{A}_1 = \{(0,0), (-1,0), (0,-1)\}, \quad \mathcal{A}_2 = \{(0,0), (1,0), (0,1), (0,2)\}.$$

Consider also the following system supported in these subsets:

$$f_i = a_{i,0} + a_{i,1}\, x_1^{-1} + a_{i,2}\, x_2^{-1}, i = 0, 1, \quad f_2 = a_{2,0} + a_{2,1}\, x_1 + a_{2,2}\, x_2 + a_{2,3}\, x_2^2.$$

with $a_{i,j} \in \mathbb{C}$.

The resultant $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2}$ is a polynomial in two sets of 3 variables and a set of 4 variables. It is multihomogeneous of multidegree $(3, 3, 1)$ and has 24 terms.

Considering the Minkowski sum $\Delta_1 + \Delta_2$ we obtain that, in this case, the only nontrivial directional sparse resultants are those corresponding to the vectors $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 0)$, $(-2, -1)$, and $(0, -1)$. Computing them together with their corresponding exponents in the Poisson formula, Theorem 6.11 shows that

$$\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = \pm a_{1,2}\, a_{1,1}^2\, a_{2,0} \prod_{i=1}^{3} f_0(\xi_i).$$

where the ξ_i's are the solutions of the system of equations $f_1 = f_2 = 0$.

Homework 6.13.

(18) Compute $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ for the Example 6.12.

Theorem 6.11 has a lot of very nice applications. For instance, one can deduce directly from there a formula for the product of the roots of a family of Laurent polynomials.

Corollary 6.14. *(*[DS15]*) For $i = 1, \ldots, n$, let $f_i \in \mathbb{C}[\boldsymbol{x}^{\pm 1}]$ be a Laurent polynomial with support contained in $\mathcal{A}_i$. Suppose that for all $v \in \mathbb{Z}^n \setminus \{0\}$, $\mathrm{Res}_{\mathcal{A}_{1,v},\ldots,\mathcal{A}_{n,v}}(f_{1,v}, \ldots, f_{n,v}) \neq 0$. Then for any $a \in \mathbb{Z}^n$,*

$$\prod_\xi \xi^{m_\xi\, a} = \pm \prod_v \mathrm{Res}_{\mathcal{A}_{1,v},\ldots,\mathcal{A}_{n,v}}(f_{1,v}, \ldots, f_{n,v})^{\langle a,v\rangle},$$

the product being over all roots $\xi \in V(f_1, \dots, f_n)$, where m_ξ denotes the multiplicity of this root.

This result makes explicit both the scalar factor and the exponents in Khovanskii's formula in [Kho99, §6, Theorem 1].

6.2. **Additivity.** From the Poisson formula, we can deduce a number of other properties for the sparse resultant. The following is the product formula for the addition of supports.

Theorem 6.15. *(*[DS15]*) Let $\mathcal{A}_0, \mathcal{A}_0', \mathcal{A}_1, \dots, \mathcal{A}_n \subset \mathbb{Z}^n$ be nonempty finite subsets and $f_0, f_0', f_1, \dots, f_n$ the general Laurent polynomials with support $\mathcal{A}_0, \mathcal{A}_0', \mathcal{A}_1, \dots, \mathcal{A}_n$, respectively. Then*

$$\begin{aligned}&\mathrm{Res}_{\mathcal{A}_0+\mathcal{A}_0',\mathcal{A}_1,\dots,\mathcal{A}_n}(f_0f_0', f_1, \dots, f_n)\\ &\qquad\qquad = \pm \mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\dots,\mathcal{A}_n}(f_0, f_1, \dots, f_n) \cdot \mathrm{Res}_{\mathcal{A}_0',\mathcal{A}_1,\dots,\mathcal{A}_n}(f_0', f_1, \dots, f_n).\end{aligned}$$

6.3. **Hidden variables vs u-resultants.** As another consequence of the Poisson formula in Theorem 6.11, one can obtain an extension to the sparse setting of the "hidden variable" technique for solving polynomial equations, which is crucial for computational purposes [CLO05, §3.5]. To state it properly, let $n \geq 1$ and, for $i = 1, \dots, n$, consider the general Laurent polynomials $f_i \in \mathbb{Z}[a_{i,\boldsymbol{\alpha}}, x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with support $\mathcal{A}_i$. Each f_i can be alternatively considered as a Laurent polynomial in the variables $\boldsymbol{x}' := \{x_1, \dots, x_{n-1}\}$ and coefficients in the ring $\mathbb{Z}[a_{i,\boldsymbol{\alpha}}, {x_n}^{\pm 1}]$. In this case, we denote it by $f_i(\boldsymbol{x}')$. The support of this Laurent polynomial is the nonempty finite subset $\varpi(\mathcal{A}_i) \subset \mathbb{Z}^{n-1}$, where $\varpi \colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection onto the first $n-1$ coordinates of $\mathbb{R}^n$. We then set

$$\mathrm{Res}^{x_n}_{\mathcal{A}_1,\dots,\mathcal{A}_n} = \mathrm{Res}_{\varpi(\mathcal{A}_1),\dots,\varpi(\mathcal{A}_n)}(f_1(\boldsymbol{x}'), \dots, f_n(\boldsymbol{x}')) \in \mathbb{C}[a_{i,\boldsymbol{\alpha}}, x_n^{\pm 1}]. \tag{6.4}$$

In other words, we "hide" the variable x_n among the coefficients of the f_i's and we consider the corresponding sparse resultant.

Theorem 6.16. *(*[DS15]*) Let notation be as above. Then, there exists $d \in \mathbb{Z}$ such that*

$$\mathrm{Res}^{x_n}_{\mathcal{A}_1,\dots,\mathcal{A}_n} = \pm x_n^d \, \mathrm{Res}_{\{\mathbf{0},\boldsymbol{e}_n\},\mathcal{A}_1,\dots,\mathcal{A}_n}(z - x_n, f_1, \dots, f_n)\big|_{z=x_n}, \tag{6.5}$$

with $\boldsymbol{e}_n = (0, \dots, 0, 1) \in \mathbb{Z}^n$.

6.4. **Work in progress.** The following results are part of an ongoing project with Martín Sombra and Gabriela Jerónimo on the study of combinatorial and computational properties of geometric sparse resultants.

6.5. **Homogeneities and extremal monomials of $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$.**
Let $\boldsymbol{\omega} = (\boldsymbol{\omega}_0, \dots, \boldsymbol{\omega}_n) \in \prod_{i=0}^n \mathbb{R}^{\mathcal{A}_i}$. For $P = \sum_{\boldsymbol{b}} p_{\boldsymbol{b}} \boldsymbol{a}^{\boldsymbol{b}} \in \mathbb{C}[a_{i,\boldsymbol{\alpha}}, \boldsymbol{\alpha} \in \mathcal{A}_i, 0 \leq i \leq n]$, we denote by $\mathrm{init}_{\boldsymbol{\omega}}(P)$ the *the initial part* of P in the direction of $\boldsymbol{\omega}$, that is,

$$\mathrm{init}_{\boldsymbol{\omega}}(P) = \sum_{\boldsymbol{b}_0} p_{\boldsymbol{b}_0} \boldsymbol{a}^{\boldsymbol{b}_0},$$

the sum being over the vectors $\boldsymbol{b}_0 \in \prod_{i=0}^n \mathbb{Z}^{\mathcal{A}_i}$ such that $\langle \boldsymbol{b}_0, \boldsymbol{\omega} \rangle = \min\{\langle \boldsymbol{b}, \boldsymbol{\omega} \rangle \mid p_{\boldsymbol{b}} \neq 0\}$.

In [Stu94, Theorem 4.1], factorization formulae for the initial part of $\mathrm{Elim}_{\boldsymbol{\mathcal{A}}}$ in the direction of $\boldsymbol{\omega}$, in terms of products of other sparse eliminants was studied for some general cases. We extend that situation to our geometric resultants and get a general and simplified picture. To do this, we proceed as follows.

Recall that for $i=0,\ldots,n$, $\Delta_i\subset\mathbb{R}^n$ is the convex hull of the set $\mathcal{A}_i$. Consider also the "lifted" polytope

$$\Delta_{i,\boldsymbol{\omega}_i}=\operatorname{conv}(\{(\boldsymbol{a},\omega_{i,\boldsymbol{a}})\mid \boldsymbol{a}\in\mathcal{A}_i\})\subset\mathbb{R}^{n+1}.$$

The vector $\boldsymbol{\omega}$ induces a *coherent decomposition* $\Delta_{\boldsymbol{\omega}}$ of $\Delta_0+\cdots+\Delta_n$ via the lower envelope of the polytope $\Delta_{0,\boldsymbol{\omega}_0}+\cdots+\Delta_{n,\boldsymbol{\omega}_n}$. For each primitive vector $\boldsymbol{v}\in\mathbb{Z}^{n+1}$ which is inner normal to a facet of the lower envelope of the lifted polytope, one can associate sets $\mathcal{A}_{i,\boldsymbol{v}}\subset\mathcal{A}_i$, $i=0,\ldots,n$, such that the lifted polytope $\operatorname{conv}(\{(\boldsymbol{a},\omega_{i,\boldsymbol{a}})\mid \boldsymbol{a}\in\mathcal{A}_{i,\boldsymbol{v}}\})$ is the face of $\Delta_{i,\boldsymbol{\omega}_i}$ in the direction of $\boldsymbol{v}$. We also set

$$f_{i,\boldsymbol{v}}=\sum_{\boldsymbol{\alpha}\in\mathcal{A}_{i,\boldsymbol{v}}}a_{i,\boldsymbol{\alpha}}\boldsymbol{x}^{\boldsymbol{\alpha}}.$$

Theorem 6.17. *With notation as above, for* $\boldsymbol{\omega}\in\prod_{i=0}^n\mathbb{R}^{\mathcal{A}_i}$,

$$\operatorname{init}_{\boldsymbol{\omega}}(\operatorname{Res}_{\boldsymbol{\mathcal{A}}})=\pm\prod_{\boldsymbol{v}}\operatorname{Res}_{\mathcal{A}_{0,\boldsymbol{v}},\ldots,\mathcal{A}_{n,\boldsymbol{v}}}\left(f_{0,\boldsymbol{v}}\ \ldots,f_{n,\boldsymbol{v}}\right),$$

where the product in the right is indexed by all the primitive vectors $\boldsymbol{v}\in\mathbb{Z}^{n+1}$ *which are inner normals to a facet of the lower envelope of* $\Delta_{0,\boldsymbol{\omega}_0}+\cdots+\Delta_{n,\boldsymbol{\omega}_n}$.

This Theorem gives a lot of extremal coefficients in $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}$, and also states the "facets of resultants are products of resultants", which is very important when it comes to the study of the Newton polytope of $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}$, see [Stu94]. These extremal "coefficients" or situations are of importance in Tropical Geometry.

6.6. **Sparse resultants under vanishing coefficients.** One of the useful applications of Theorem 6.17 to give a formula for computing $\operatorname{Res}_{\boldsymbol{\mathcal{A}}}\left(\widetilde{f}_0,\ldots,\widetilde{f}_n\right)$, with $\widetilde{f}_i$ being the general Laurent polynomial with support in a subset $\widetilde{\mathcal{A}}_i\subset\mathcal{A}_i$, $i=0,\ldots,n$. A partial approach to this problem has been done in [Min03], who considered the case when $\widetilde{\mathcal{A}}_i=\mathcal{A}_i$ for all but one i.

To deal with this problem, we consider the vector $\boldsymbol{\omega}\in\prod_{i=0}^n\mathbb{Z}^{\mathcal{A}_i}$ given by

$$\omega_{i,a}=\begin{cases}-1 & \text{if } \boldsymbol{a}\in\widetilde{\mathcal{A}}_i,\\ 0 & \text{otherwise.}\end{cases}\tag{6.6}$$

As before, there is a decomposition $\Delta_{\boldsymbol{\omega}}$ of the Minkowski sum $\Delta_0+\cdots+\Delta_n$ made by taking the lower envelope of the lifted polytope $\Delta_{0,\boldsymbol{\omega}}+\cdots+\Delta_{n,\boldsymbol{\omega}}\subset\mathbb{R}^{n+1}$. Note that $\boldsymbol{v}_0=(0,\ldots,0,1)$ is one of the inner normals of the facets of the lower envelope, associated to the data $(\widetilde{\mathcal{A}}_0,\ldots,\widetilde{\mathcal{A}}_n)$. All the other facets of this lower envelope are in correspondence to inner vectors $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_N$ having its last coordinate positive and first n coordinates not identically zero. We write $\boldsymbol{v}_i=(\boldsymbol{v}_{i,1},v_{i,n+1})$, $i=1,\ldots,N$, with $\boldsymbol{v}_{i,1}\in\mathbb{Z}^n\setminus\{\boldsymbol{0}\}$, and $v_{i,n+1}\in\mathbb{Z}_{>0}$.

The decomposition induced by (6.6) is said to be *relevant* if for all $i=1,\ldots,N$, the associated decomposition $\left(\mathcal{A}_{0,\boldsymbol{v}_i},\ldots,\mathcal{A}_{n,\boldsymbol{v}_i}\right)$ has either more than one essential subfamily, or the unique essential subfamily $\{\mathcal{A}_{j,\boldsymbol{v}_i}\}_{j\in J}$ satisfies $\mathcal{A}_{j,\boldsymbol{v}_i}\subset\widetilde{\mathcal{A}_i}$ for all $j\in J$.

Theorem 6.18. *With notation as above, we have that* $\operatorname{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}\left(\widetilde{F}_0,\ldots,\widetilde{F}_n\right)\neq 0$ *if and only if the induced coherent mixed decomposition is relevant. If this is the case,* *then*

$$\operatorname{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}\left(\widetilde{f}_0,\ldots,\widetilde{f}_n\right) = \pm \operatorname{Res}_{\widetilde{\mathcal{A}}_0,\ldots,\widetilde{\mathcal{A}}_n} \cdot \prod_{i=1}^{N} \operatorname{Res}_{\mathcal{A}_{0,\boldsymbol{v}_i},\ldots,\mathcal{A}_{n,\boldsymbol{v}_i}}\left(f_{0,\boldsymbol{v}}\ \ldots, f_{n,\boldsymbol{v}}\right),.$$

Note that in principle $\operatorname{Res}_{\mathcal{A}_{0,\boldsymbol{v}_i},\ldots,\mathcal{A}_{n,\boldsymbol{v}_i}}$ may depend on more variables than those indexed by the points in $\widetilde{\mathcal{A}}_0,\ldots,\widetilde{\mathcal{A}}_n$, but the condition of the system being relevant actually implies that this cannot happen.

Example 6.19. Set $n = 1$, and $\mathcal{A}_0 = \{0, 1, \ldots, A\}$, $\mathcal{A}_1 = \{0, 1, \ldots, B\}$, with A, $B \in \mathbb{Z}_{>0}$. If we pick $\widetilde{\mathcal{A}}_0 = \{0\}$, $\widetilde{\mathcal{A}}_1 = \{B\}$, then the decomposition induced by this choice has two cells: $(\{0\}, \mathcal{A}_1)$ and $(\mathcal{A}_0, \{B\})$, corresponding to the inner normals in $\mathbb{Z}^2$ given by $\boldsymbol{v}_1 = (1, B)$ and $\boldsymbol{v}_2 = (-1, A)$ respectively. The decomposition is relevant in this case, as the unique essential subfamilies here are $\{0\}$ and $\{B\}$ respectively. Theorem 6.18 states that $\operatorname{Res}_{\mathcal{A}_0,\mathcal{A}_1}(a_{0,0}, a_{1,B})$ is actually equal to

$$\operatorname{Res}_{\{0\},\{B\}} \cdot \operatorname{Res}_{\{0\},\mathcal{A}_1} \cdot \operatorname{Res}_{\mathcal{A}_0,\{B\}} = \pm a_{0,0}^{B} \cdot a_{1,B}^{A},$$

which can be easily verified.

On the other hand, if we chose $\widetilde{\mathcal{A}}_0 = \widetilde{\mathcal{A}}_1 = \{0\}$, then the cell decomposition may have one or two cells depending whether $A = B$ or not. In the first case, we have only one cell which is $(\mathcal{A}_0, \mathcal{A}_1)$ corresponding to the (not necessarily primitive) inner normal in $\mathbb{Z}^2$ given by $(-2, A + B)$. Note that this decomposition is not relevant as the only essential subfamily induced by this vector is $\{\mathcal{A}_0, \mathcal{A}_1\}$, which is not contained in $(\widetilde{\mathcal{A}}_0, \widetilde{\mathcal{A}}_1)$, so we have that $\operatorname{Res}_{\mathcal{A}_0,\mathcal{A}_1}(a_{0,0}, a_{1,0}) = 0$.

If $A \neq B$, suppose w.l.o.g. that $A < B$, then we have two cells in the decomposition: $(\{0\}, \mathcal{A}_1)$ induced by $(-1, B)$,and $(\mathcal{A}_0, \{B\})$ induced by $(-1, A)$. This decomposition is also not relevant as in the second cell we have that the only essential subfamily is the singleton $\{B\}$ which is not contained in $\widetilde{\mathcal{A}}_1$. Hence, $\operatorname{Res}_{\mathcal{A}_0,\mathcal{A}_1}(a_{0,0}, a_{1,0}) = 0$ also in this case, as it is easy to verify.

6.7. **Isobarism of the geometric sparse resultant.** For $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^\times)^n$, set

$$f_{i,\boldsymbol{\lambda}} = f_i(\lambda_1 x_1, \ldots, \lambda_n x_n) = \sum_{\boldsymbol{\alpha}\in\mathcal{A}_i} a_{i,\boldsymbol{\alpha}}(\lambda_1 x_1)^{\alpha_1} \ldots (\lambda_n x_n)^{\alpha_n}.$$

It is well-known that there exist $A_1, \ldots, A_n \in \mathbb{N}$ such that

$$\operatorname{Res}_{\boldsymbol{\mathcal{A}}}\left(f_{0,\boldsymbol{\lambda}}, \ldots, f_{n,\boldsymbol{\lambda}}\right) = \lambda_1^{A_1} \cdots \lambda_n^{A_n} \operatorname{Res}_{\boldsymbol{\mathcal{A}}}(f_0, \ldots, f_n), \tag{6.7}$$

see for instance [GKZ94]. We compute explicitely these numbers by using again lifted functions and convex geometry, as above:

Theorem 6.20. *Suppose that* $\Delta_i \subset (\mathbb{R}_{\geq 0})^n$. *For* $i, j = 0, \ldots, n$, *and for* $j = 0, \ldots, n$, *consider the convex hull*

$$\Delta_{j,i,} = \operatorname{conv}\big(\{(\boldsymbol{x},\, x_i), (\boldsymbol{x},\, 0) \mid \boldsymbol{x} \in \Delta_j\}\big) \subset \mathbb{R}^{n+1}.$$

Then the exponent A_i *in* (6.7) *is equal to* $\operatorname{MV}_{n+1}(\Delta_{0,i}, \ldots, \Delta_{n,i})$.

Example 6.21. Let $d_0, \ldots, d_n \in \mathbb{N}$, and set $\mathcal{A}_i = \{(a_1, \ldots, a_n) \in \mathbb{N}^n : \sum_{j=1}^{n} a_j \leq d_i\}$. Denote by $\Delta^n \subset \mathbb{R}^n$ the standard simplex, defined as the convex hull of $\mathbf{0} \in \mathbb{R}^n$ and the vectors in the standard basis of this space. With the above notation, we have that $\Delta_i = d_i\, \Delta^n$. Denote with $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1}\}$ the canonical basis of $\mathbb{R}^{n+1}$. It is easy to see that the polytope $\Delta_{j,i}$ turns out to be the convex hull in $\mathbb{R}^{n+1}$ of the vectors $\mathbf{0}$, $d_j\boldsymbol{e}_1, \ldots, d_j\boldsymbol{e}_n$, $d_j(\boldsymbol{e}_i + \boldsymbol{e}_{n+1})$. This polytope can be transformed straightforwardly into $d_j\,\Delta^{n+1} \subset \mathbb{R}^{n+1}$ with a linear transformation of determinant 1, hence has the same volume of the latter. Note that the same linear transformation converts each $\Delta_{j,i\mu_j}$ into $d_j\,\Delta^{n+1}$, $j = 0, \ldots, n$. This fact plus elementary properties of the mixed volume already, implies that, for all $i = 0, \ldots, n$,

$$A_i = d_0 \cdots d_n.$$

So, we recover the well-known result of *isobarism* of resultants in the homogeneous case, which was presented in the previous section.

6.8. **Work in progress: Macaulay style formulae for sparse resultants.** We are currently working on determinantal formulae for computing sparse resultants. We hope to simplify and generalize the formulae in [D'A02] in the sense that the determinant of the resultant matrices now are going to be multiples of the sparse resultant, the assumption that the whole family of supports $\{\mathcal{A}_0, \ldots, \mathcal{A}_n\}$ is essential will be dropped, and the construction of both the resultant matrix and the extraneous factor is going to be given with a simpler recursion.

7. Open Problems

(1) Same problem with the height of the sparse resultant as in the homogeneous case. In [DS15] it is shown that

$$\mathrm{h}(\mathrm{Res}_{\boldsymbol{\mathcal{A}}}) \leq \sum_{i=0}^{n} \mathrm{MV}_M(\Delta_0, \ldots, \Delta_{i-1}, \Delta_{i+1}, \ldots, \Delta_n) \log(\#\mathcal{A}_i).$$

Is this bound sharp? Can you find nontrivial coefficients which grow as the size of the supports gets larger?

(2) Can you compute $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ as the "determinant of a complex"? In [GKZ94] there are several complexes whose determinant give the resultant in very smooth cases where all the supports are n-dimensional and generate the whole lattice $\mathbb{Z}^n$. Do the same complexes exist in all the cases where $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ is not equal to 1? Is there a version of this for the geometric sparse resultant? Does this new formulation simplify the presentation given in [GKZ94]? This question was raised by David Cox.

(3) All the construction for the sparse resultant $\mathrm{Res}_{\boldsymbol{\mathcal{A}}}$ developed here (and detailed in [DS15]) is made by using the complex numbers $\mathbb{C}$ as a base field. One can extend striaghtforwardly all the results in that paper to resultants over fields of characteristic zero, but the situation with fields of positive characteristic is not very clear yet, due to the fact that standard results on toric varieties used heavily in all the proofs of [DS15] are known to be valid in characteristic zero only. In addition, the multiplicity of the map $\pi_1|_V : V \to \pi_1(V)$ in characteristic zero is a very "geometric" situation, namely the number of points in the fiber of a generic point in the image, and we get a whole description of this degree via this interpretation. In positive characteristic, extra care must be taken to handle with this situation. In the dense case, the theory in any field (and any commutative ring) is very well extended and known (cf. [Jou97]).

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