Sparse Resultants of Toric Cycles

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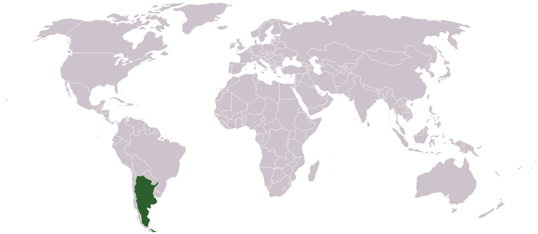
Computational Algebra,Algebraic Geometry & Applications



A Conference in honor of Alicia Dickenstein

Buenos Aires, Argentina, August 1–3 2016

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Resultants

Important for both algorithmic and complexity aspects of polynomial system solving

Example

Given

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We get the i-th coordinates of the roots via

$$\mathrm{Res}_{d_1,\ldots,d_n}(f_1^i, \ldots, f_n^i) = Res_{1,d_1,\ldots,d_n}(t - x_i, f_1, \ldots, f_n)$$

Does the same happen in the sparse world?

$$\mathrm{Res}_{\pi_i(\mathcal{A}_1),\ldots,\pi_i(\mathcal{A}_n)}(f_1^i,\ldots,f_n^i) = Res_{\{\mathbf{0},\mathbf{e}_i\},\mathcal{A}_1,\ldots,\mathcal{A}_n}(t-x_i,f_1,\ldots,f_n) \quad ?$$

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The sparse resultant of $f_0, \ldots, f_n$ should be

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- The defining equation of

$$\begin{array}{ccc} W = \{(\mathbf{c}_{i,\mathbf{a}}, \mathbf{x}) : \ f_i = 0\} & \subset & \mathbb{P}^{m_0} \times \ldots \times \mathbb{P}^{m_n} \times (\mathbb{C}^\times)^n \\ \downarrow & & \downarrow \pi \\ \pi(W) & \subset & \mathbb{P}^{m_0} \times \ldots \times \mathbb{P}^{m_n} \end{array}$$

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- $\pi(W)$ does not have codimension 1

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- ...
- **But you lose some geometrical invariants**

Our motivation

(**D**- Galligo-Sombra, AMJ 2014)

Give a close (and elegant) formula for

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$$\mathrm{Res}_{\{\mathbf{0},\mathbf{e}_i\},\mathcal{A}_1,\mathcal{A}_2}(t - x_i, f_1(x_1), f_2(x_2)) = f_i(t)$$

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$\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}$ is the defining equation of the direct image $\pi_* W$

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- Classically: $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = \det(c_{ij})$
- With the new definition, $\mathrm{Res}_{\mathcal{A}_0,\mathcal{A}_1,\mathcal{A}_2} = \det(c_{ij})^4$

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- $\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}$ is not irreducible anymore but the power of an irreducible element (the old resultant)
- $\deg_{c_{i,a}}\left(\mathrm{Res}_{\mathcal{A}_0,\dots,\mathcal{A}_n}\right) = MV(Q_0,\dots,\widehat{Q_i},\dots,Q_n)$
$$Q_i = \text{convex hull}(\mathcal{A}_i)$$

Algebra meets Geometry

(if $\pi(W)$ has codimension one)

$$\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}(f_0,\ldots,,f_n)=0$$
$$\Updownarrow$$
$$\exists p\in X_{\mathcal{A}}|\ f_0(p)=\ldots=f_n(p)=0$$

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$X_{\mathcal{A}}$ is a toric variety defined by $\mathcal{A}_0,\ldots,\mathcal{A}_n$ with respect to $(\mathbb{C}^\times)^n$

Toric Varieties vs Toric Cycles

$$\mathcal{A}_i = \{a_{i,0}, \dots, a_{i,m_i}\} \subset \mathbb{Z}^n$$

$$\begin{array}{rcl} \varphi_{\mathcal{A}} \colon (\mathbb{C}^\times)^n & \to & \mathbb{P}^{m_0} \times \dots \mathbb{P}^{m_n} \\ \xi & \mapsto & \left(\left(\xi^{a_{0,0}} : \dots : \xi^{a_{0,m_0}}\right), \dots, \left(\xi^{a_{n,0}} : \dots : \xi^{a_{n,m_n}}\right)\right) \end{array}$$

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- $X_{\mathcal{A}} = \overline{\varphi_{\mathcal{A}}\left((\mathbb{C}^\times)^n\right)}$ is a multiprojective toric subvariety
- $Z_{\mathcal{A}} = (\varphi_{\mathcal{A}})_*(\mathbb{C}^\times)^n$ is a toric cycle

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Our definition coincides with the *resultant* (in the sense of Rémond) of the cycle $Z_{\mathcal{A}}$

Rémond's resultants of cycles

(Élimination multihomogène LNM 1752, 2001)

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$$\mathrm{Res}_X(F_0, \ldots, F_d) = \mathrm{Res}_{X \cdot Z(F_0)}(F_1, \ldots, F_d)$$

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- $\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n} = \prod_v \mathrm{Res}_{\mathcal{A}_{1,v},\ldots,\mathcal{A}_{n,v}}^{-h_{\mathcal{A}_0}(v)} \cdot \left(\prod_\xi f_0(\xi)^{m_\xi}\right)$

$$\xi \in V(f_1, \ldots, f_n)$$

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- "hidden variables":

$$\mathrm{Res}_{\pi_i(\mathcal{A}_1),\ldots,\pi_i(\mathcal{A}_n)}(f_1^i, \ldots, f_n^i) = \pm x_i^d\, \mathrm{Res}_{\{\mathbf{0},\mathbf{e_i}\},\mathcal{A}_1,\ldots,\mathcal{A}_\mathbf{n}}(t - x_i, f_1, \ldots, f_n)\big|_{t=x_i}$$

Initial forms

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$$\mathrm{init}_{\omega}(\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}) = \pm \prod_{\mathbf{v}} \mathrm{Res}_{\mathcal{A}_{0,\mathbf{v}},\ldots,\mathcal{A}_{n,\mathbf{v}}}(f_{0,\mathbf{v}} \ \ldots, f_{n,\mathbf{v}}),$$

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the product is over all primitive $\mathbf{v} \in \mathbb{Z}^{n+1}$ inner normals to a facet of the lower envelope of the "lifted" polytopes $Q_{0,\omega_0} + \cdots + Q_{n,\omega_n}$

Vanishing Coefficients

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$$\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}(\tilde{f}_0,\ldots,\tilde{f}_n) = \pm \mathrm{Res}_{\tilde{\mathcal{A}}_0,\ldots,\tilde{\mathcal{A}}_n} \cdot \prod_{\mathbf{v}} \mathrm{Res}_{\mathcal{A}_{0,\mathbf{v}},\ldots,\mathcal{A}_{n,\mathbf{v}}}$$

iff the CMD is "relevant", otherwise it vanishes identically

More Homogeneities

(**D**-Jerónimo-Sombra)

$$\mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}\big(f_0(\lambda_1 x_1,\ldots,\lambda_n x_n),\ldots,f_n(\lambda_1 x_1,\ldots,\lambda_n x_n)\big) = \lambda_1^{A_1}\cdots\lambda_n^{A_n}\cdot \mathrm{Res}_{\mathcal{A}_0,\ldots,\mathcal{A}_n}$$

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where

$$A_i = MV_{\mathbb{Z}^{n+1}}(Q_{0,i,\mu_0},\ldots,Q_{n,i,\mu_n}) + \sum_{j=0}^{n}\mu_j MV_{\mathbb{Z}^n}(Q_0,\ldots,Q_{i-1},Q_{i+1},\ldots,Q_n)$$

with $\mu_j \ll 0$, and $Q_{j,i,\mu_j} =$ convex hull$\big(\{(\mathbf{x},\,x_i),(\mathbf{x},\,\mu_i) \mid \mathbf{x}\in Q_j\}\big) \subset \mathbb{R}^{n+1}$

"Sylvester type" formulae

(**D**-Jeronimo-Sombra)

Work in Progress!

$$\mathcal{R}(f_0, f_1, f_2) = \det \begin{bmatrix} -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 & 0 \\ -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -b_1 & -b_3 & 0 & a_1 & a_3 & 0 & 0 & 0 & 0 \\ 0 & -b_0 & -b_2 & 0 & a_0 & a_2 & 0 & 0 & 0 & 0 \\ -c_4 & -c_5 & -c_8 & 0 & 0 & 0 & a_1 & 0 & a_3 & 0 \\ -c_1 & -c_3 & -c_7 & 0 & 0 & 0 & a_0 & a_1 & a_2 & a_3 \\ -c_0 & -c_2 & -c_6 & 0 & 0 & 0 & 0 & a_0 & 0 & a_2 \\ 0 & 0 & 0 & -c_4 & -c_5 & -c_8 & b_1 & 0 & b_3 & 0 \\ 0 & 0 & 0 & -c_1 & -c_3 & -c_7 & b_0 & b_1 & b_2 & b_3 \\ 0 & 0 & 0 & -c_0 & -c_2 & -c_6 & 0 & b_0 & 0 & b_2 \end{bmatrix}$$

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