Rational Plane Curves Parameterizable by Conics

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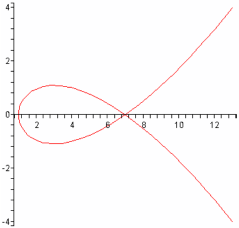
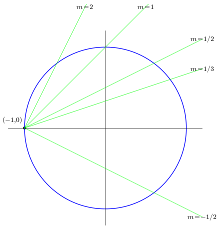
Rational Plane Curves

- $\mathbb{K}$ is an algebraically closed field
- $\phi : \mathbb{P}^1_{\mathbb{K}} \to \mathbb{P}^2_{\mathbb{K}}$ polynomial parameterization
- $\phi(\underline{t}) = (u_1(\underline{t}) : u_2(\underline{t}) : u_3(\underline{t}))$ homogeneous with $\gcd(u_i(\underline{t})) = 1$
- $\mathrm{C} := \overline{\mathrm{Im}(\phi)}$



The (computational) Implicitization Problem

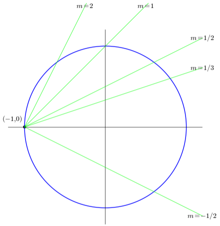
Given $(u_1(\underline{t}) : u_2(\underline{t}) : u_3(\underline{t}))$, compute $E(X_1, X_2, X_3) \in \mathbb{K}[X_1, X_2, X_3]$ such that $\mathrm{C} = \{E(X_1, X_2, X_3) = 0\}$



$$\underline{u}(\underline{t}) = (2t_1 t_2, t_1^2 - t_2^2, t_1^2 + t_2^2) \quad \to \quad E(X_1, X_2, X_3) = X_1^2 + X_2^2 - X_3^2$$

The (computational) Inversion Problem

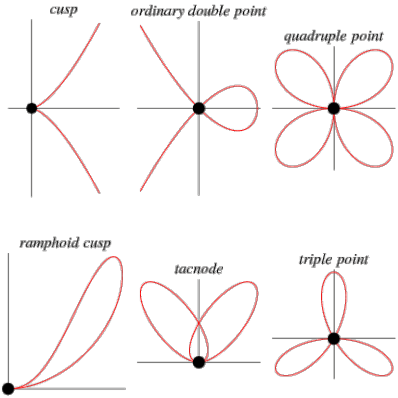
Given $(u_1(\underline{t}) : u_2(\underline{t}) : u_3(\underline{t}))$, compute $F_1(X_1, X_2, X_3), F_2(X_1, X_2, X_3) \in \mathbb{K}[X_1, X_2, X_3]$ such that $\mathrm{C} \overset{(F_1:F_2)}{\dashrightarrow} \mathbb{P}^1_{\mathbb{K}}$ is the inverse of $\mathbb{P}^1_{\mathbb{K}} \overset{\underline{u}(\underline{t})}{\to} \mathrm{C}$



$$\underline{u}(\underline{t}) = (2t_1t_2, t_1^2 - t_2^2, t_1^2 + t_2^2) \quad \to \quad (F_1, F_2) = (X_2 + X_3, X_1)$$

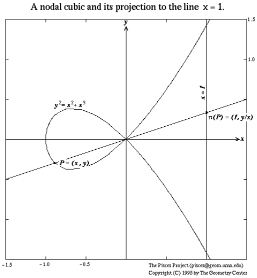
(meta)-Fact

If Sing(C) is "small", then both the implicitization/inversion problem gets easier



Curves Parameterizables by lines

$\mathbb{C}$ has one singular point of multiplicity equal to $\deg(\mathbb{C}) - 1$



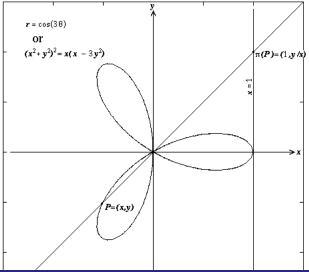
By Bézout's theorem, any line passing through the singular point intersects $\mathbb{C}$ in another single point
The pencil of lines passing through the singular point produces a parameterization $\underline{u}(\underline{t})$ of the curve

Curves Parameterizables by lines

Assume $p = (0:0:1)$ is the singular point of $\mathcal{C}$
For any $a(\underline{t})$, $b(\underline{t})$ of degrees $d-1$, d respectively such that $\gcd(a,b) = 1$, we have

- $\underline{u}(\underline{t}) = (t_1 a(\underline{t}), t_2 a(\underline{t}), b(\underline{t}))$
- $E(\underline{X}) = b(X_1, X_2) - X_3 a(X_1, X_2)$
- $(F_1, F_2) = (X_1, X_2)$

A curve of degree 4 with a triple point



The Geometry of $\mathcal{C}$ around its singular point

Write
$$a(X_1, X_2) = \mathbf{c}\prod_{j=1}^{\tau}(\mathbf{d}_j X_2 - \mathbf{e}_j X_1)^{\nu_j}$$

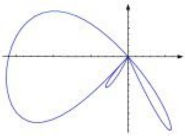
$\mathbf{c} \in \mathbb{K} \setminus \{0\}$, $(\mathbf{d}_j : \mathbf{e}_j) \neq (\mathbf{d}_k : \mathbf{e}_k)$ if $j \neq k$, $\nu_j \in \mathbb{N}$

- There are τ different branches of $\mathcal{C}$ passing through p
- The tangent to the branch $\gamma_j(\underline{t})$ at $\underline{t}_j$ is the line $\mathbf{d}_j X_2 - \mathbf{e}_j X_1 = 0$
- Different branches have different tangents (no tacnodes)
- The order of contact of $\mathcal{C}$ with $\mathbf{d}_j X_2 - \mathbf{e}_j X_1 = 0$ at p is equal to $\nu_j - 1$

Bonus Track

A set of minimal generators of the Rees Algebra associated to the parameterization given by $\underline{u}(\underline{t})$ is very easy to get in terms of $a(\underline{t}), b(\underline{t})$

- Cox, Hoffman, Wang 2008
- Busé 2009



Curves Parameterizable by Conics

(joint work with Teresa Cortadellas)

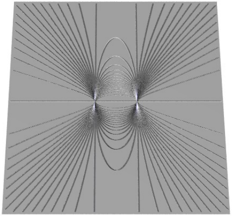
- A curve is parameterizable by lines iff there exist a birational morphism $\mathcal{C} \overset{(F_1:F_2)}{\dashrightarrow} \mathbb{P}^1_{\mathbb{K}}$ such that $\deg(F_1, F_2) = 1$
- A curve is parameterizable by conics iff there exist a birational morphism $\mathcal{C} \overset{(F_1:F_2)}{\dashrightarrow} \mathbb{P}^1_{\mathbb{K}}$ such that $\deg(F_1, F_2) = 2$

Pencils of Conics

$F_1(\underline{X}),\ F_2(\underline{X}) \in \mathbb{K}[X_1, X_2, X_3]$ homogeneous of degree 2
C is parameterizable by (F_1, F_2) if and only if the system

$$\begin{cases} t_1 F_2(\underline{X}) - t_2 F_1(\underline{X}) & = & 0 \\ \\ E(\underline{X}) & = & 0 \end{cases}$$

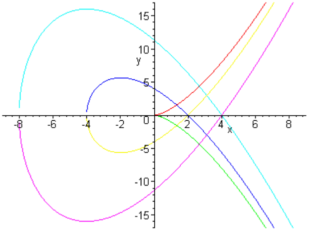
has $2d - 1$ solutions in $\mathbb{P}^2_{\mathbb{K}}$ and one $(\underline{u}(\underline{t}))$ in $\mathbb{P}^2_{\mathbb{K}(\underline{t})}$



Singularities

$$\mathrm{Sing}(\mathbb{C}) \subset V(F_1, F_2)$$

Curves parameterizable by conics have at most 4 singular points



Lines vs Conics

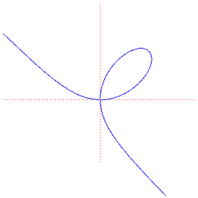
Any cubic is parameterizable both by lines and conics!

$$\underline{u}(\underline{t}) = (3t_1t_2^2, 3t_1^2t_2, t_1^3 + t_2^2)$$

$$(F_1(\underline{X}), F_2(\underline{X})) = \begin{cases} (X_2, X_1) \\ (X_1^2, X_2^2 - 3X_2X_3) \end{cases}$$

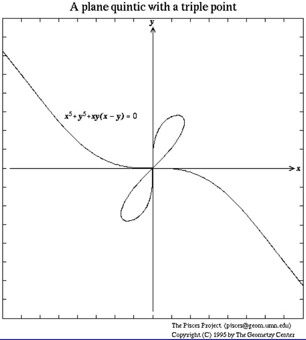
$$E(\underline{X}) = X_1^3 + X_2^3 - 3X_1X_2X_3$$

Folium of Descartes



Lines vs Conics (Cortadellas - D')

- If $\deg(\mathrm{C}) > 3$, then the curve cannot be parameterized by both lines and conics.
- More generally, if C is parameterizable by forms of degree d and d', then $d + d' \geq \deg(C)$
- Generically, C is parameterizable by forms of degree $\deg(\mathrm{C}) - 2$



Curves Parameterizable by Conics

Assume $(0:0:1) \in V(F_1, F_2)$ and write

$$t_1 F_2(\underline{X}) - t_2 F_1(\underline{X}) = l_1(\underline{t})X_1X_2 + l_2(\underline{t})X_1X_3 + l_3(\underline{t})X_2X_3 + l_4(\underline{t})X_1^2 + l_5(\underline{t})X_2^2$$

Pick $a(\underline{t}),\ b(\underline{t}) \in \mathbb{K}[\underline{t}]_d$ without common factors

Theorem (Cortadellas - D')

If $F_1(\underline{X})\,F_2(\underline{X})$ depend on X_3 then there is a parameterization of a rational curve $\mathbb{C}$ parameterizable by (F_1, F_2) given by

$$\begin{cases} u_1(\underline{t}) &= -a(\underline{t})(l_1(\underline{t})a(\underline{t}) + l_2(\underline{t})b(\underline{t})) \\ u_2(\underline{t}) &= -b(\underline{t})(l_1(\underline{t})a(\underline{t}) + l_2(\underline{t})b(\underline{t})) \\ u_3(\underline{t}) &= l_1(\underline{t})a(\underline{t})b(\underline{t}) + l_4(\underline{t})a(\underline{t})^2 + l_5(\underline{t})b(\underline{t})^2 \end{cases}$$

The implicit equation of $\mathbb{C}$ is given by

$$a(F_1(\underline{X}), F_2(\underline{X}))X_2 - b(F_1(\underline{X}), F_2(\underline{X}))X_1$$

or an irreducible (computable) factor of it.

Comparison at level of Syzygies

- C is parameterizable by lines if and only if for any proper parameterization $\underline{u}(\underline{t})$ of C, there is an element in $\mathrm{Syz}(\underline{u}(\underline{t}))$ of degree one
- If C is parameterizable by conics then for any proper parameterization $\underline{u}(\underline{t})$ of C, the minimal degree of a nonzero element in $\mathrm{Syz}(\underline{u}(\underline{t}))$ is $\lfloor \deg(\mathrm{C})/2 \rfloor$

($\Longrightarrow$ "moderate" singularities)

Example

- $(F_1(\underline{X}), F_2(\underline{X})) = (X_2^2, X_1X_3 - X_2^2)$
- $V(F_1, F_2) = \{(0:0:1), (1:0:0)\}$, both double points

$\underline{u}(\underline{t}) = (t_1^{2k} + t_1^{2k-1}t_2, t_1^k t_2^k, t_2^{2k})$ is a parameterization of $\mathrm{C} = V((X_1X_2 - X_2^2)^k - X_2^{2k-1}X_3)$, with inverse (F_1, F_2)

$\mathrm{Syz}(\underline{u}(\underline{t})) = \langle t_1^k X_3 - t_2^k X_2,\ t_2^k X_1 - t_1^k X_2 - t_1^{k-1} t_2 X_2 \rangle$

Example (continuation)

- $(F_1(\underline{X}), F_2(\underline{X})) = (X_2^2, X_1X_3 - X_2^2)$
- $V(F_1, F_2) = \{(0:0:1), (1:0:0)\}$, both double points

$\underline{u}(\underline{t}) = (t_1^{2k+1} + t_1^{2k}t_2, t_1^k t_2^{k+1}, t_2^{2k+1})$ is a parameterization of $\mathrm{C} = V(X_3(X_1X_2 - X_2^2)^k - X_2^{2k+1})$, with inverse (F_1, F_2)

$\mathrm{Syz}(\underline{u}(\underline{t})) = \langle t_1^k X_3 - t_2^k X_2,\ t_2^{k+1}X_1 - t_1^{k+1}X_2 - t_1^k t_2 X_2\rangle$

The Geometry of C around its singular points

Set $\mathcal{F} = \{(1:0:0), (0:1:0), (0:0:1)\}$ and consider the Cremona Transformation

$$\begin{array}{rccc} \tau : & \mathbb{P}^2_{\mathbb{K}} & \dashrightarrow & \mathbb{P}^2_{\mathbb{K}} \\ & (x_1 : x_2 : x_3) & \mapsto & (x_2x_3 : x_1x_3 : x_1x_2) \end{array}$$

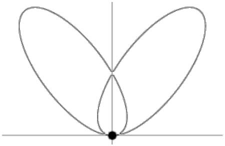
Theorem (Cortadellas - D')

If $\mathcal{F} \subset V(F_1, F_2)$ then C is parameterizable by (F_1, F_2) iff $\tau(\mathrm{C})$ is parameterizable by lines with unique singular point not in $\mathcal{F}$
Reciprocally, for any $\tilde{\mathrm{C}}$ parameterizable by lines with unique singular point not in $\mathcal{F}$, $\tau(\tilde{\mathrm{C}})$ is parameterizable by conics

The Geometry of $\mathbb{C}$ around its singular points

Theorem (Cortadellas - D')

- If $|V(F_1, F_2)| = 4$, then $\mathbb{C}$ looks locally like parameterizable by lines around each of these points (no tacnodes, etc.)
- If $|V(F_1, F_2)| = 3$, then around the double point there will be a tangent to several folds. In a neighbourhood of any of the other two points, $\mathbb{C}$ is like before.



The Geometry of $\mathcal{C}$ around its singular points

If $V(F_1, F_2) = \{(0:1:0), (0:0:1)\}$ we will consider the following quadratic transformation

$$\begin{array}{cccc} \tau' : & \mathbb{P}^2_{\mathbb{K}} & \dashrightarrow & \mathbb{P}^2_{\mathbb{K}} \\ & (x_1 : x_2 : x_3) & \mapsto & (x_1x_2 : x_1^2 : x_2x_3) \end{array}$$

- τ' is not defined on $\{(0:1:0), (0:0:1)\}$
- τ' is birational, indeed $\tau' \circ \tau' = id$
- the line $X_1 = 0$ is not in $\mathrm{Im}(\tau')$ (only the point $(0:0:1)$ is)
- τ' can be regarded as a limit of the Cremona transformation τ when $(1:0:0) \to (0:0:1)$

Theorem (Cortadellas - D')

If $V(F_1, F_2) = \{(0:1:0), (0:0:1)\}$ then $\mathcal{C}$ is parameterizable by (F_1, F_2) iff $\tau'(\mathcal{C})$ is parameterizable by lines with unique singular point in $\{X_3 = 0\}$

Reciprocally, for any $\tilde{\mathcal{C}}$ parameterizable by lines with unique singular point in $\{X_3 = 0\}$, $\tau'(\tilde{\mathcal{C}})$ is parameterizable by conics

The Geometry of $\mathbb{C}$ around its singular points

If $V(F_1, F_2) = \{(0:1:0), (0:0:1)\}$, the geometry of $\mathbb{C}$ around these points will depend on whether they are both double or one of them is triple.

- Around a single point of $V(F_1, F_2)$, there is a similar behaviour as in the monoid case
- If the point is double or triple, then the transformation will merge the tangents of the different branches but separate the curvatures (the singularity becomes more complicated!)

The Geometry of $\mathcal{C}$ around its singular points

If $V(F_1, F_2) = \{(0:0:1)\}$ we will consider

$$\begin{array}{cccc} \tau'' : & \mathbb{P}^2_{\mathbb{K}} & \dashrightarrow & \mathbb{P}^2_{\mathbb{K}} \\ & (x_1 : x_2 : x_3) & \mapsto & (x_1^2 : x_2^2 + x_1x_3 : x_1x_2) \end{array}$$

- τ'' is not defined on $\{(0:0:1)\}$
- $\tau''^{-1} = (x_1^2 : x_1x_3 : x_1x_2 - x_3^2)$, i.e. τ'' is birational
- the line $X_1 = 0$ is not in $\mathrm{Im}(\tau'')$ (only the point $(0:1:0)$ is)
- $\tau''(\{X_1 = 0\}) = (0:1:0)$
- τ'' can be regarded as $\lim \tau'$ when $(0:1:0) \to (0:0:1)$

Theorem (Cortadellas - D')

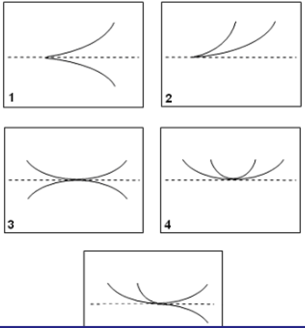
If $V(F_1, F_2) = \{(0:0:1)\}$ then C is parameterizable by (F_1, F_2) iff $\tau''^{-1}(\mathrm{C})$ is parameterizable by lines with $(0:0:1)$ being its only singular point

Reciprocally, for any $\tilde{\mathrm{C}}$ parameterizable by lines with $(0:0:1)$ being its only singular point, $\tau''(\tilde{\mathrm{C}})$ is a curve parameterizable by conics with only one singularity in $(0:0:1)$

The Geometry of C around its singular points

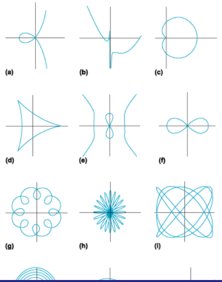
Around $(0:0:1)$, C has all the branches coming from a singularity parameterizable by lines plus all the images of points of the form $(0:\alpha:\beta) \in \mathrm{C}$, $\alpha \neq 0$.

At $(0:0:1)$, τ'' merges tangent lines and curvature, but separates forms of third degree or more



Conclusion

- Curves parameterizable by conics are essentially the image of curves parameterizable by lines via a quadratic transformation of $\mathbb{P}^2_{\mathbb{K}}$
- They have at most 4 singularities
- The geometry of the curve (and the quadratic transformation) depends on the number of singularities and their multiplicities in $V(F_1, F_2)$, the larger the number the simpler the structure



Bonus Track (Cortadellas - D')

A set of minimal generators of the Rees Algebra associated to any proper parameterization of a curve parameterizable by conics is very easy to get in terms of
$\{F_1(\underline{X}),\, F_2(\underline{X}),\, a(\underline{t}),\, b(\underline{t})\}$

Moltes Gràcies!