Al-Khayyām's Scientific Revision of Algebra

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Abstract
Geometers in medieval Islam found algebra to be a useful tool in solving problems, but the Greek restriction of "number" to positive integers, adhered to by al-Khayyām (and presumably others), is at odds with the more liberal inclusion of fractions and irrational roots in traditional numerical Arabic algebra. By regarding the known and unknown numbers of traditional algebra as the homogeneous measures of continuous magnitudes (and not as magnitudes themselves), al-Khayyām was able to give a rigorous foundation for the art. This revision helps us understand al-Khayyām's main work on algebra in the context of geometrical and numerical problem solving, and it leads us to a reassessment of his generalized notion of "number" explained in his commentary on Euclid.

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1. Introduction

Arabic algebra began as a practical method of arithmetical problem solving.[2] At least as far back as the early ninth century merchants, surveyors, accountants, and even the judges charged with the division of estates were said to make use of the technique. What made algebra versatile is that it could be used to solve any linear or quadratic problem, provided that the parameters be converted to numbers. For instance, one worked with numerical lengths and areas when solving mensuration problems.

But algebra was not just applied to practical problems. It also turned out to be effective for solving problems in theoretical geometry that could not be worked out with straight edge and compass. This is accomplished by first setting up and simplifying a cubic equation that satisfies the conditions of the problem. The terms of the equation are then reinterpreted in the context of a new diagram in which two conic sections intersect, and the solution to the original problem is a line extending to this intersection. This technique was already well established when ʻUmar al-Khayyām (1048-ca.1131) conceived a plan to systematize it. There are only twenty-five types of simplified cubic and lower degree equations,[3] so a book listing all their solutions could serve the needs of future geometers. To meet the requirements of theoretical geometry he would also need to give algebra a rigorous foundation and provide proofs to all his solutions.

Al-Khayyām first announced his plan in *Risāla fī taqsīm rubʻ al-dāʻira* (*Treatise on the division of a quadrant of a circle*, henceforth *Quadrant*). In this book he solves a geometry problem via algebra. The quartic equation he sets up simplifies to $x^3 + 200x = 20x^2 + 2000$, and this is solved by intersecting a semicircle with a hyperbola. In the middle of the solution, between the simplified equation and the construction, al-Khayyām gives a summary of his views on algebra. Later he wrote his famous *Risāla fī'l-barāhīn ʻalā masā'il al-jabr wa'l-muqābala* (*Treatise on the proofs of algebra problems*, henceforth *Algebra*). Here he realizes his project by solving all twenty-five simplified equations of degree three and less.

[2] By "practical" I mean that the method was taught to people in preparation for careers requiring calculation. While merchants may have been unlikely to solve quadratic equations on the job, they learned algebra as part of their education.

[3] There are 25 types because coefficients were required to be positive. For example, the equations $x^3 + 2x = 4$ and $x^3 + 4 = 2x$ were considered to be of different kinds. See §2 below.

Al-Khayyām's *Algebra* has been well known to modern historians since the publication of Franz Woepcke's edition and translation in 1851. Since then the classification of equations and the solutions to the 14 irreducible[4] cubic cases have received ample and deserved attention. But the way al-Khayyām made practical calculation rigorous and his revision of the structure of algebraic solutions has been largely overlooked.

Algebraists who wrote practical textbooks admitted fractions and irrational roots as numbers, but al-Khayyām remained faithful to the Greek notion that numbers consist only of positive integers (possibly excluding 1). With such a restriction his algebraic powers cannot be unknown numbers. Instead, al-Khayyām regarded them as the abstraction of "quantity" common to continuous magnitudes.

This distinction between discrete numbers and continuous magnitudes necessitates a reworking of the third stage in the solutions of problems, that of solving the simplified equation. In traditional arithmetical algebra all unknowns are numbers, so the solutions of the six simplified equations of degrees 1 and 2 are given solely as numerical recipes. Now al-Khayyām finds the need to give two separate solutions, where possible, to each equation: one if the original unknown is a number, and another if it is a magnitude.

I begin this article with a review of the structure of practical Arabic algebra to put al-Khayyām's work in perspective. After presenting a translation of a problem from Abū Kāmil's *On the Pentagon and Decagon*, solved in the traditional manner, I give al-Khayyām's problem from *Quadrant*. This is followed by an analysis of al-Khayyām's program. By observing how he manipulates his terms and from comments he makes in both works we can determine the nature of al-Khayyām's powers of the unknown and his modification of the structure of algebraic solutions. A corollary of this analysis is a revision of our understanding of al-Khayyām's generalized concept of "number".

[4] "Irreducible" equations are those with a constant term, and thus do not reduce to one of lower degree.

2. Arabic algebra before al-Khayyām

Modern accounts of Arabic algebra, especially those which describe al-Khwārizmī's early ninth century *Kitāb al-jabr wa'l-muqābala*[5] ("Book of Algebra"), tend to focus on the solutions to the six simplified quadratic and linear equations and their geometric proofs. Before putting al-Khayyām's *Algebra* in perspective I must put al-Khwārizmī's six equations in the perspective of practical algebraic problem solving.

Muḥammad ibn Mūsā al-Khwārizmī was asked by the Caliph al-Ma'mūn (reigned 813-833 C.E.) to write a book on algebra which "encompasses the fine and important parts of its calculations that people constantly require in cases of their inheritance, their legacies, their partition, their law-suits, and their trade, and in all their dealings with one another, such as the surveying of land, the digging of canals, mensuration, and other various aspects and kinds are concerned."[6] Practical mathematics does not concern itself with philosophical notions that limit numbers to positive integers, so the concept of "number" for these people is any positive quantity that arises in calculation, including fractions and irrational roots. Measurements of length, area, weight, or time are always numerical. This is what al-Khwārizmī meant when he wrote in the beginning of his *Algebra* "When I considered what people generally want in calculating, I found that it is always a number."[7]

Al-Khwārizmī's treatment of algebra, like many which followed, is divided into two parts. The first part contains an explanation of the powers of the unknown, the classification and solutions of the six simplified equations (here with proofs), and rules for operating with roots and polynomials. The second part is a collection of thirty-nine worked-out problems. The purpose of the rules in the first part is to train the student to solve the problems in the second part.[8]

[5] The title given by Rosen (1831) and later historians is *Kitāb al-mukhtaṣar fī ḥisāb al-jabr wa'l-muqābala* ("Brief Book on Calculation by Algebra"). Now Rashed claims that the original title was simply *Kitāb al-jabr wa'l-muqābala* ("Book of Algebra") [al-Khwārizmī 2007, 9].

[6] [al-Khwārizmī 2007, 95]. Translation adapted from [Gutas 1998, 113], which in turn was adapted from Rosen's translation [al-Khwārizmī 1831, 3].

[7] [al-Khwārizmī 2007, 97.1; al-Khwārizmī 1831, 5]. Rosen's translation.

[8] The Oxford MS has 40 problems, but one of them was probably added later. After the worked-out problems al-Khwārizmī gives a short section on the rule of three and mensuration, followed by a long collection of worked-out inheritance problems.

The name given to the first degree unknown in Arabic algebra is *shay'* ("thing"), and sometimes also *jidhr* ("root"). Its square is called *māl* (literally "sum of money", "property").[9] Units were counted in "dirhams" (singular *dirham*, a silver coin), *min al-ʻadad* ("in number") or as *āḥād* ("units"), and frequently the appellation was dropped. Our earliest text with higher powers is Qusṭā ibn Lūqā's translation of Diophantus' *Arithmetica*, written a few decades later. The third power is *kaʻb* ("cube"), and higher powers were written as combinations of *māl* and *kaʻb*, such as *māl māl* for x^4 and *māl kaʻb* for x^5. A typical equation, from al-Khwārizmī's problem (4), is "twenty-one things and two thirds of a thing less two *māl*s and a sixth equals a hundred and two *māl*s less twenty things."[10] In modern notation this is $21\frac{2}{3}x - 2\frac{1}{6}x^2 = 100 + 2x^2 - 20x$. An algebraic notation emerged in the Maghreb around the 12th c. CE, but the texts of earlier algebraists, including those by al-Khwārizmī, Abū Kāmil, and al-Khayyām, are entirely rhetorical.

The solution to a problem by medieval Arabic algebra can be broken down into three stages:

> Stage 1: An unknown number is named in terms of the algebraic powers (usually as a "thing"). The operations or conditions specified in the enunciation are then applied to arrive at an equation.
>
> Stage 2: The equation is simplified to a standard form. The six standard equations of the first two degrees are described below.
>
> Stage 3: The solution to the simplified equation is found using the prescribed procedure.

Like in medieval Europe, negative numbers and zero and were not acknowledged in Arabic mathematics. Because the solutions to simplified equations take the numbers (coefficients) of the terms as parameters, no term can be subtracted. This yields the following six types of equation: (1) $ax^2 = bx$, (2) $ax^2 = c$, (3) $bx = c$, (4) $ax^2 + bx = c$, (5) $ax^2 + c = bx$, and (6) $bx + c = ax^2$. Al-Khwārizmī and later algebraists show how to solve each of

[9] Because there is no good single-word English translation of *māl*, and because the word was used in algebra in a technical sense unrelated to its quotidian meaning, I leave it untranslated. I write its plural with the English suffix: *māl*s.
[10] [al-Khwārizmī 2007, 163.9].

the six types with a numerical rule. Al-Khwārizmī's solution to the sample type 4 equation "a half *māl* and five roots equal twenty-eight dirhams" ($\frac{1}{2}x^2+5x=28$) is:

> So you want to complete your *māl*, so that it becomes whole, which is that you double it. So double it and double everything you have which is equated with it. So it yields: a *māl* and ten roots equals fifty-six dirhams. So halve the roots,[11] so it yields five. So multiply it by itself, so it yields twenty-five. So add it to the fifty-six, so it yields eighty-one. So take its root, which is nine. So subtract from it half the roots, which is five. So there remains four, which is the root of the *māl* that you wanted, and the *māl* is sixteen and its half is eight.[12]

Beginning with al-Khwārizmī and his contemporary Ibn Turk, a few of the more scientifically minded algebraists include geometric proofs of the validity of these procedures.[13] There the *māl* is represented by a square, and its side is a "thing". The geometric magnitudes in these proofs merely *represent* the unknown numbers. They are not identical with them.[14]

Below is a sample worked-out problem from Abū Kāmil's *On the Pentagon and Decagon*, which is included as part of his *Book of Algebra* (late 9th c.). I chose this problem because it is an application of algebra to geometry done in the traditional manner. Keep in mind that the vast majority of problems solved in Arabic algebra are arithmetic questions.

[11] Here "the roots" is short for "the number of the roots", or what we call the coefficient of the first degree term.

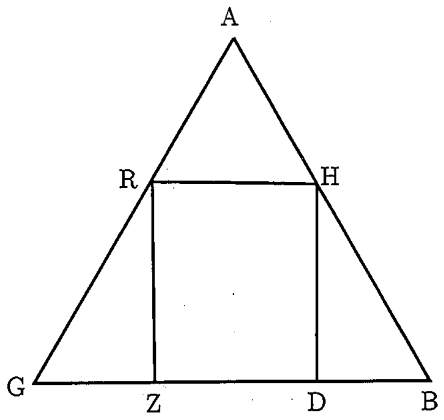
[12] [al-Khwārizmī 2007, 103.10].

[13] The following authors give geometric proofs: Al-Khwārizmī (#41 [M3]), Ibn Turk (#59 [M1]), Thābit ibn Qurra (#103 [M19]), Abū Kāmil (#124 [M1]) (all four from the 9th c.), al-Karajī (*al-Fakhrī*, early 11th c., #309 [M2]), and al-Samaw'al (12th c., #487 [M1]). Some algebraists who either give arithmetical proofs, or no proofs at all, are: 'Alī al-Sulamī (10th c., #267 [M1]), al-Karajī (*al-Kāfī*, early 11th c., #309 [M1]), Ibn al-Yāsamīn (late 12th c., #521 [M3]), Ibn Badr (13th c.?, #587 [M1]), al-Fārisī (d. ca. 1320, #674 [M2]), Ibn al-Bannā' (d. 1321, #696, [M6, M8, M11]), al-Umawī (14th c., #931 [M2]), Ibn al-Hā'im (1387, #783 [M13]), al-Kāshī (d. 1436, #802 [M1]), Sibṭ al-Māridīnī (late 15th c., #873 [M10]), al-Qalaṣādī (d. 1486, #865 [M3], [M7]), Ibn Ghāzī (d. 1513, #913 [M2]), and al-'Āmilī (ca. 1600, #1058 [M1]). The numbering of authors and works is taken from [Rosenfeld & Ihsanoğlu 2003].

[14] In some of Abū Kāmil's proofs the *māl* is represented by a line. Also, beginning about the 12th c. there was a shift away from geometric proof in favor of arithmetical proofs.

[Enunciation]
So if [someone] said to you: a triangle with equal sides and angles. Each of its sides is ten, and in it is a rectangle of area ten. What is the height of the rectangle?

[Stage 1]
An example of this is that we make the triangle triangle *ABG*, and the rectangle in it rectangle *HRDZ*. So if we wanted to know how much line *HD* is, which is the height of the rectangle, we make it a thing [x]. So line *BD* is the root of a third of a *māl* [$\sqrt{\frac{1}{3}x^2}$], and likewise line *ZG* is the root of a third of a *māl*. So this leaves line *DZ* as ten less the root of a *māl* and a third [$10-\sqrt{1\frac{1}{3}x^2}$]. So you multiply it by line *HD*, which is a thing, so it yields: ten things less the root of a *māl māl* and a third of a *māl māl* equals ten dirhams [$10x-\sqrt{1\frac{1}{3}x^4}=10$].



[Stage 2]
So restore the things [by] the root of a *māl māl* and a third of a *māl māl* and add its same to the dirhams. So you get: ten dirhams and the root of a *māl māl* and a third of a *māl māl* equals ten things [$10+\sqrt{1\frac{1}{3}x^4}=10x$].

[Stage 3]
So return [the root of a *māl māl* and a third of a *māl māl*][15] to the root of a *māl māl*, which is a *māl*, which is that you multiply it by the root of three fourths.

[15] The MS has "everything you have" for my restored text in brackets.

> And you multiply everything you have by the root of three fourths. So it yields: a *māl* and the root of seventy five dirhams equals the root of seventy five *māl*s [$x^2+\sqrt{75}=\sqrt{75x^2}$].
>
> So halve the root of seventy-five *māl*s, so it yields the root of eighteen and a half and a fourth. So multiply it by itself: it yields eighteen and a half and a fourth. So cast away from it the root of seventy-five, leaving eighteen and a half and a fourth less the root of seventy five. So you take the root. So what results, you added to it the root of eighteen and a half and a fourth. So the sum is the height of the rectangle, which is line *HD*.[16]

In stage 3 Abū Kāmil uses the procedure for the type 5 equation to find that the height of the rectangle is the number $\sqrt{18\frac{3}{8}-\sqrt{75}}+\sqrt{18\frac{3}{8}}$. Many of the rules given in the first part of his book are applied in this problem: operations on polynomials in stage 1, operations on square roots in stages 1 and 3, and the solution to the simplified equation in stage 3. Like many algebraists, Abū Kāmil does not explain *al-jabr* (restoration) and *al-muqābala* (confrontation), the steps used when necessary in stage 2.

The rules and proofs for the six equations are just one part of practical Arabic algebra. The rules are given so the practitioner or student can complete stage 3 in the solution to a problem, and proofs are provided in some books as an added measure of rigor. To many modern readers these proofs are the highlight of Arabic algebra, and so they have received the bulk of attention. This imbalance has been reinforced by al-Khayyām's *Algebra*, where our author gives only the rules, constructions, and proofs for the simplified equations of degree three and less. Because al-Khayyām wrote for mathematicians already familiar with practical algebra, he had no need to teach the rules for manipulating polynomials and roots, nor to include a collection of worked-out problems illustrating the method of algebra.

3. Al-Khayyām's worked out problem

Medieval Arabic mathematicians took up the challenge of geometrical calculation that they inherited from the Greeks. While classical Greek geometers wrote their "calculations" in purely geometric terms, there was a trend beginning in late antiquity to posit the existence of a unit segment and

[16] Problem (13) [Abū Kāmil 1986, 147.16].

to express calculations with language borrowed from arithmetic.[17] For example, Archimedes related the volume of a sphere to that of a cone: "Every sphere is four times a cone having a base equal to the greatest circle of the <circles> in the sphere, and, <as> height, the radius of the sphere."[18] The Banū Mūsā (9th c.), by contrast, expressed the volume of a sphere as "the product of half of its diameter by a third of its surface area".[19] The surface of a sphere is curved, so this product cannot be a geometric object given in position. It is instead an abstract "quantity" resulting from the multiplication of two other "quantities".

This arithmetization facilitated the application of algebra to geometry. Al-Khayyām himself relates details of the history of these applications in both of his treatises.[20] In the late ninth century al-Māhānī attempted to use algebra to solve the problem of cutting a sphere by a plane so that the two parts are in a given ratio, from Archimedes' *On the Sphere and Cylinder* II.4. He reduced the problem to an equation involving "cubes, *māl*s and numbers",[21] but he could not solve it. In the next century Abū Ja'far al-Khāzin made a breakthrough. He reinterpreted the "cube", "*māl*", and "number" as geometric magnitudes, and solved the equation as a line extending to the intersection of two conic sections in a new diagram. Later geometers were then able to solve assorted solid problems with conic sections via similar transformations through algebra.

Al-Khayyām, too, found algebra to be an expedient tool for solving a geometry problem in his *Quadrant*. He begins by reducing the main problem, that of dividing the quadrant of a circle into two parts satisfying a certain condition, to a problem of constructing a particular right triangle. This he solves with algebra.

17 [Cuomo 2000, 180; Rashed 1996, 370, 372; Rashed & Vahabzadeh 1999, 8/2000, 8; Brentjes 2008, 452].

18 [Netz 2004, 148], his translation. That the cone is a third of the cylinder with the same base and height was already well known.

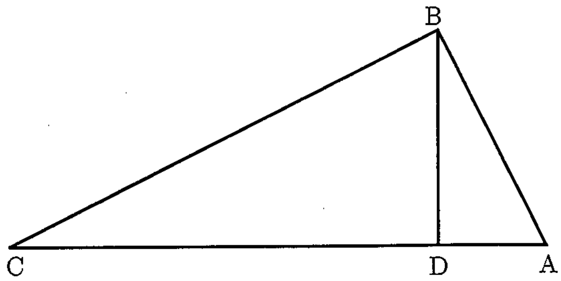
19 [Rashed 1996, 113.8]. They had just proven in the previous proposition that the surface of a hemisphere is double the area of its great circle.

20 [Rashed & Vahabzadeh 1999, 117.11, 227.9, 255.4/2000, 111, 160, 173].

21 [Rashed & Vahabzadeh 1999, 117.13/2000, 111].

[Enunciation]

So suppose [we have] triangle *ABC* and angle *B* is right. And we draw from point *B* a perpendicular *BD* to *AC*. And we suppose that side *AB* with the perpendicular *BD* together are equal to *AC*…[22]



[Stage 1]

And we suppose that the line *AD* is rational in length: let it be ten. And we suppose that *BD* is a thing, and we multiply it by itself: it yields a *māl*. And we multiply ten by itself: it yields a hundred. And we sum them: they yield a hundred and a *māl*, which is the square on *AB*, as is demonstrated in 47 of I.[23] And since the ratio of *AC* to *AB* is as the ratio of *AB* to *AD*, because of the similarity of the two triangles *ABC*, *ABD*, it follows that the product of *AC* by *AD* is equal to the square on *AB*. So if we divided the square on *AB*, which is a hundred in number and a *māl*, by *AD*, which is ten, [then] the result of the division is ten in number and a tenth of a *māl*, which is *AC*. And we supposed that *AC* is equal to the sum of *AB*, *BD*. So the sum of *AB*, *BD* is ten in number and a tenth of a *māl*. We subtracted from it *BD*, which is the thing, leaving ten in number and a tenth of a *māl* less a thing, which is *AB*. So we multiply it by itself. It results in: a hundred in number and three *māl*s and a tenth of a tenth of a *māl māl* less twenty things, and less a fifth of a cube equals a hundred in number and a *māl* [$100+3x^2+\frac{1}{10}\frac{1}{10}x^4-20x-\frac{1}{5}x^3=100+x^2$].

[22] [Rashed & Vahabzadeh 1999, 247.5/2000, 169] (my translation).
[23] Euclid's *Elements* I.47, the Pythagorean theorem.

[Stage 2]
So one restores and confronts and collapses,[24] leaving two *māl*s and a tenth of a tenth of a *māl māl* equals twenty things and a fifth of a cube [$2x^2 + \frac{1}{10}\frac{1}{10}x^4 = 20x + \frac{1}{5}x^3$]. So one divides everything by the thing, so it results in the four lesser types [i.e. powers] in this ratio. So the result of the division is: a tenth of a tenth of a cube and two things equals a fifth of a *māl* and twenty in number [$\frac{1}{10}\frac{1}{10}x^3 + 2x = \frac{1}{5}x^2 + 20$].

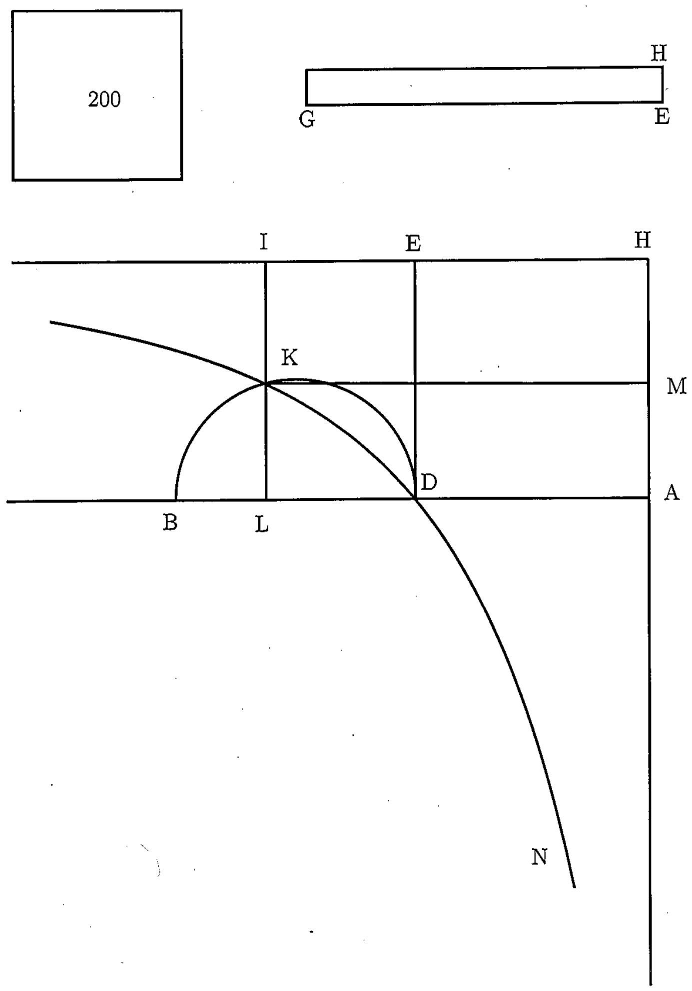
So one completes a tenth of a tenth of a cube by multiplying by a hundred. Likewise we multiply all of the types by a hundred. So it results in: a cube and two hundred things equals twenty *māl*s and two thousand in number [$x^3 + 200x = 20x^2 + 2000$].[25]

[After a long digression in which he outlines his algebraic program, al-Khayyām returns to work out stage 3. Note the switch to geometric language at this point.]

[Stage 3]
We return to our problem, after reviewing these preliminaries: To find a cube, which together with two hundred of its sides, is equal to twenty squares of its side with two thousand in number. We suppose line *AB* is equal to the number of squares, which is twenty, and line *EG* is two hundred, and line *EH* one. Thus surface *HG* is two hundred. And we make a square equal to surface *HG* as is shown in Proposition 14 of Book II. And let the side of that square be equal to *AH*, where *AH* is perpendicular to *AB*, and it [*AH*] is the root of two hundred. And *AD* is the result of the division if one divides the number by the number of roots, which is ten, since the number is two thousand and the number of roots is two hundred. And if one divides the two thousand by two hundred it results in ten. And *DB* likewise is ten.…[26]

[24] I explain the phrase "restoration and confrontation" (*al-jabr wa'l-muqābala*) in [Oaks & Alkhateeb 2007]. In al-Khayyām's equation this entails "restoring" the $100 + 3x^2 + \frac{1}{10}\frac{1}{10}x^4$ by the subtracted $20x + \frac{1}{5}x^3$, and "confronting" the $3x^2$ on the left with the x^2 on the right. I have not seen the term "collapse" (*qāḍa*) used in the context of simplifying an equation in any other book. It may refer to "collapsing" the two 100's on either side.
[25] [Rashed & Vahabzadeh 1999, 247.13/2000, 169] (my translation). This last step, multiplying each term by 100, would have traditionally been considered to be part of stage 3. But in al-Khayyām's algebra it fits more naturally in stage 2.
[26] [Rashed & Vahabzadeh 1999, 257.14/2000, 174] (my translation).

[From here al-Khayyām sets up the semicircle *BKD* and the hyperbola *KDN*, and he proves that line *AL* (= *MK*) solves the equation.]



There is no noticeable difference in the ways Abū Kāmil and al-Khayyām carry out stages 1 and 2 in their solutions. Both assign numerical values to at least one line, and the length of another line is made a "thing". The equations they set up are simplified to a standard form, with no subtracted terms and only one term for each power.[27] Abū Kāmil's unknowns are numbers which can take non-integer values, so the similarities in the solutions suggest that al-Khayyām's unknowns are numerical as well. By observing how al-Khayyām manipulates his algebraic terms, and from the comments he gives in both treatises, we will see that in practice he does indeed work in the same numerical system as earlier algebraists, but that he regards these numbers to be the dimensionless abstraction of "quantity" shared by continuous magnitudes.

4. Stages 1 & 2: Working with abstract "quantity"

Al-Khayyām remained faithful to Aristotle regarding philosophy of mathematics. He cited Aristotle in his criticism of Euclid's and Ibn al-Haytham's admission of motion to geometry,[28] and he adhered to the Stagirite's partition of the genus of "quantity" into discrete and continuous. On this Aristotle wrote "Of quantities some are discrete, others continuous… Discrete are number and language; continuous are lines, surfaces, bodies, and also, besides these, time and place."[29] Al-Khayyām makes the same distinction in his *Commentary*:

> And when they found number to be of the same genus as magnitude, because both of them were altogether divided under the genus of quantity, they looked as well for this notion in the magnitudes. Then they found in them together with these two divisions another division. That is, the magnitudes are not made up of indivisible parts, and there is no definite end to their division, as there is with number.[30]

Al-Khayyām's identification of "number" with positive integers puts restrictions on his numerical solutions to equations in the *Algebra*. For the first compound quadratic equation he writes:

[27] Higher powers were common in practical algebra. In other problems Abū Kāmil set up a higher degree equation which reduces to one of lower degree. For example, in problem (51) he sets up the equation $x^4 + 12\frac{1}{4}x^2 - 7x^3 = 2x^3$, which reduces to $x^2 + 12\frac{1}{4} = 9x$. [Abū Kāmil 1986, 96.20]

[28] In his *Commentary* [Rashed & Vahabzadeh 1999, 311/2000, 219].

[29] *Categories* 4b20-24, translated in [Aristotle 1963, 12]. "Number" is discrete because it encompasses only positive integers, and "language" consists of individual syllables.

[30] [Rashed & Vahabzadeh 1999, 343.5/2000, 235] (their translation).

> A *māl* and ten roots equals thirty-nine in number [$x^2+10x=39$]. So multiply half the roots by its same, and add the result to the number, and subtract from the root of the outcome half the roots. So the remainder is the root of the *māl*.
>
> Numerically, these two conditions are necessary: the first one of them, that the number of the roots be an even number, so that it may have a moiety; and the second, that the sum of the square of half the number of the roots and the number be a square number. For otherwise the problem would then be impossible numerically. But geometrically, not a thing at all pertaining to its problems will be impossible.[31]

With this framework in mind, we can now ask how al-Khayyām's "number", "thing", *māl*, "cube", *māl māl*, etc. relate to the genus of "quantity". We already know that they cannot be unknown numbers because the powers can represent the measures of continuous magnitudes in the solutions to geometry problems. In fact, al-Khayyām implies that they are somehow associated with continuous magnitudes in the *Algebra*. In the beginning of the book, just before naming the powers of the unknown, he writes: "And magnitudes are a continuous quantity, of which there are four: the line, the surface, the solid and time, as it is mentioned in a general way in the *Categories* and in detail in *First Philosophy*…"[32] Specifically what kinds of objects al-Khayyām's powers are emerges from the way he operates on them, and from comments he makes about their nature.

Al-Khayyām begins his algebraic solution in *Quadrant* by making line *AD* "ten". Of course "ten" only *measures* the line, it is not identical with it. He calls *BD* a "thing", and later line *AC* turns out to be "ten in number and a tenth of a *māl*". So "number", "thing", and *māl* can all measure lines. Also in stage 1 the square on *AB* is calculated as "a hundred and a *māl*", so "number" and *māl* can also measure surfaces. Later on the square on *AB* is also found to be "a hundred in number and three *māl*s and a tenth of a tenth of a *māl māl* less twenty things, and less a fifth of a cube", an aggregation of the first five powers. The ways these powers are combined shows that they

[31] [Rashed & Vahabzadeh 1999, 137.5/2000, 120] (1st paragraph: my translation, 2nd paragraph: their translation). Woepcke and Rashed & Djebbar suggest that al-Khayyām commits an error here. It is true that an integer solution can exist if the number of roots is odd. But al-Khayyām may be speaking of the *calculation* of the answer, not the *existence* of the answer. [Woepcke 1851, 42; Rashed & Djebbar 1981, 107].

[32] [Rashed & Vahabzadeh 1999, 121.14/2000, 113] (their translation). After this quote he argues against Aristotle's "place" as a continuous magnitude.

are homogeneous, and that no term is tied to a particular dimension. Homogeneity is also evident from the fact that the powers are proportional: "…the ratio of the number to the roots is as the ratio of the roots to the *māl*s, and as the ratio of the *māl*s to the cubes, and as the ratio of the cubes to the *māl*s *māl*s, as far as one wishes to go."[33]

We glean further insight from al-Khayyām's explanation of the fourth degree term: "What is here called by the algebraists a *māl māl* is something imagined in the continuous magnitudes, and which cannot exist by any means in sensible things."[34] By "sensible things" he means lines, planes, and cubes. (Later we will see also that the zero degree term "number" is likewise "abstracted by the mind".) A *māl māl* (and the other higher powers) is a continuous measure, but it does not correspond in a natural way to any particular dimension. Couple this with the fact that "number", "thing" and *māl* can each measure magnitudes of different dimensions, and we see that *all* the powers of the unknown must be "imagined in the continuous magnitudes" during the manipulations in stages 1 and 2. They are the abstraction of "quantity", stripped of dimension, which continuous magnitudes share in this genus.

By regarding the algebraic unknowns as abstract "quantity", al-Khayyām gives a rigorous foundation to the number system used by practical mathematicians. He writes in his *Commentary*:

> …those who make calculations, I mean those who make measurements, often speak of one half of the unit, of one third thereof, and of other parts, although the unit be indivisible. But they do not mean by this a true absolute unit whereof true numbers are composed. On the contrary, they mean by this an assumed unit which, in their opinion, is divisible. Then they act in whatsoever way they please in the management of the magnitudes in accordance with that divisible unit and in accordance with the numbers composed thereof; and they often speak of the root of five, of the root of the root of ten, and of other things which they constantly do in the course of their discussions, and in their constructions and their measurements. But they only mean by this a 'five' composed of divisible units, as we have mentioned.[35]

[33] [Rashed & Vahabzadeh 1999, 123.1/2000, 113] (my translation).
[34] [Rashed & Vahabzadeh 1999, 249.20/2000, 170] (adapted from their translation). "Sensible things" is Rashed's and Vahabzadeh's translation of *wujūh* (sing. *wajh*). This term is translated in their French edition as *individus*.
[35] [Rashed & Vahabzadeh 1999, 379.15/2000, 253] (their translation).

Divisible units belong to continuous magnitudes, so al-Khayyām suggests in this quotation that earlier algebraists, Abū Kāmil included, had unknowingly worked with the same abstract "quantities" which he used in his algebraic solution.

5. Stage 3: dual solutions of quadratic equations

Al-Khayyām asserted in *Quadrant* that algebra belongs wholly to geometry:

> And those who think that algebra is a means to extract unknown numbers think the unthinkable; therefore you must not pay attention to those who judge appearances and are of a different opinion: on the contrary, algebra is something geometrical which is demonstrated in Book II of the *Elements*, in Propositions 5 and 6 thereof.[36]

But later, when he wrote his *Algebra*, he decided that the art *could* be used as a tool for solving arithmetic problems. The different natures of numbers and geometric magnitudes compelled him to give two solutions to each simplified equation: a numerical rule if the original problem is in arithmetic, and a geometric construction if it is in geometry. Al-Khayyām gives arithmetical and geometric solutions to all equations of degrees 1 and 2 and for reducible cubic equations. (A cubic equation is "reducible" if it has no constant term, and thus can be reduced to one of lower degree.)[37] For the fourteen irreducible cubic equations he gives only the geometric solutions because he could not solve them numerically. For these equations he writes "when the subject of the problem is an absolute number, it was not feasible either for us or for any one of those concerned with the art—and possibly someone else will come to know it after us…"[38]

The equations with dual solutions follow a general pattern. The rule for the numerical solution is given first, accompanied by a geometric proof that the rule is correct. Then comes the geometric solution with its proof. (Some of these elements are missing for each equation.)

[36] [Rashed & Vahabzadeh 1999, 251.14/2000, 171] (their translation).

[37] Three-dimensional geometric solutions are given for the reducible cubic equations, and for their numerical solutions the reader is referred back to the solution to the corresponding quadratic equation. Marco Panza also wrote about these dual solutions in [Panza 2007, 127ff].

[38] [Rashed & Vahabzadeh 1999, 125.5/2000, 114] (their translation). Numerical solutions were first found by del Ferro and Tartaglia in the 16th c.

The presence of dual solutions raises the question of the degree of abstraction of al-Khayyām's unknowns. In an equation like "a *māl* and ten roots equals thirty-nine in number" the value of the "root" can be found as a magnitude or as a number. Could al-Khayyām have regarded it as being a magnitude *in general*, possessing a nature which rises beyond the discrete and the continuous to encompass both kinds of quantity? Such a modern interpretation is unlikely. Because the arithmetical unit is distinct from the continuous unit, al-Khayyām probably viewed the abstract "numbers" attached to continuous quantity as capable of *representing* the discrete numbers of arithmetic. In this way the continuous "root" can represent an unknown positive integer.

6. Stage 3: Geometric solutions of irreducible cubic equations

For each of the fourteen irreducible cubic equations solved in the *Algebra* the geometric construction flows seamlessly into its proof. The construction itself utilizes only the "numbers" (coefficients) of the terms of the equation, while in the proof the powers are reinterpreted as the measures of geometric magnitudes of specific dimensions. Al-Khayyām describes this transition in the interlude between stages 2 and 3 in his solution in *Quadrant*. He first explains that only powers up to the cube can be part of such a reinterpretation. "And as to things which are used by the algebraists, and which exist in sensible things and in continuous magnitudes, they are fourfold: number, thing, *māl*, and cube." While "number" is a bit tricky, the other three powers are associated with their natural geometric counterparts:

> As for number, number is taken as abstracted by the mind from material things; and it does not exist in sensible things, since number is a universal intelligible thing which cannot exist except when individuated by material things. And as to the thing, its rank (*manzila*) in relation to continuous magnitudes is the rank of the straight line. And as to the *māl*, its rank is that of the rectangular equilateral quadrangle whose side is that straight line to which the expression thing is applied. And the cube is a solid which is contained by six equal rectangular equilateral square surfaces…And as to the *māl māl*, which according to the algebraists results from the product of the *māl* by itself, it does not have any

> meaning in the continuous magnitudes, for the square being a surface, how could it be multiplied by itself?…[39]

Al-Khayyām proves that the line *AL* satisfies the equation in *Quadrant* in the same way he proves his solutions in his *Algebra*: by comparing different volumes. Each term in the equation "a cube and two hundred things equals twenty *māl*s and two thousand in number" ($x^3 + 200x = 20x^2 + 2000$) is reinterpreted as the measure of a three dimensional rectangular solid. The "cube" in the equation becomes the cube on *AL*. The "two hundred things" is the product of the square on *LI* by *AL*, where the area of the square is 200. The "twenty *māl*s" is the product of the square on *AL* by *AB*, where *AB* is twenty. Last is the "two thousand", which is the square on *LI* by *AD*, or 200 by 10. Homogeneity is maintained by making the "numbers" (coefficients) of the first and second degree terms a plane and a line respectively, and by making the 2000 the measure of a solid.

The reinterpretation of the zero degree term "number" (*'adad*) among the different equations in the *Algebra* depends on the degree of the equation. In the proofs of the constructions for cubic equations "number" measures a solid, while for quadratic equations it measures a plane. Al-Khayyām writes in the *Algebra* "And whenever we say in this treatise *a number is equal to a surface*, we mean by *number* a rectangular surface one of whose sides is a unit while the second is a line equal in measure to the given number…", and "And whenever we say *a number is equal to a solid*, we will mean by *number* a rectangular parallelepipedal solid whose base is the square of the unit and whose height is equal to the supposed number."[40] So while "thing", *māl*, and "cube" are always manifested as 1, 2, and 3-dimensional objects respectively in the proofs, the dimension of "number" changes with the degree of the equation. This is why it must be "abstracted by the mind from material things", just like the *māl māl*.

Al-Khayyām was understandably uncomfortable about including "number" with the other "geometrical" degrees. On the one hand it is necessary to reinterpret "number" as a geometric object in the proofs. But on the other hand it does not correspond naturally to any particular degree. His ambivalence is apparent in both treatises. We saw above in *Quadrant* that he

[39] [Rashed & Vahabzadeh 1999, 249.27/2000, 171] (adapted from their translation). Rashed & Vahabzadeh translate *manzila* as "position" in both French and English.

[40] [Rashed & Vahabzadeh 1999, 131.10, 133.7/2000, 117, 118] (their translation).

includes "number" in the list of things "which exist in sensible things", but then he writes that it "does not exist in sensible things". In his *Algebra* he writes that only three degrees "fall within magnitudes": "root" (i.e. "thing"), *māl*, and "*cube*" (identified with side, surface, and body respectively). But immediately after he writes "…what is found in the works of the algebraists with respect to these equations between the four geometrical degrees, I mean absolute numbers, sides, squares, and cubes".[41]

7. Al-Khayyām's concept of continuous "number"

Al-Khayyām takes up the issue of the compounding (multiplication) of ratios in the third part of his *Commentary*. It is here that he explains his concept of "number" in the context of ratios of magnitudes. He begins his proof that $A:B$ compounded with $B:C$ is $A:C$ by writing:

> We assume the unit, and we set its ratio to the magnitude *G* equal to the ratio of *A* to *B*. And the magnitude *G* should not be regarded as being a line, or a surface, or a solid, or a time. On the contrary, it should be regarded as being abstracted by the mind from these adjunct characters and as being attached to number…[42]

Al-Khayyām describes *G* with the same kind of language he used for the algebraic terms *māl māl* and "number" (and by homogeneity all the algebraic powers). All belong to the abstract scale of measurement shared by continuous magnitudes. This level of generality is necessary in the proof because al-Khayyām needs to work in a dimensionless system. At one point he multiplies the magnitude *D*, which is homogeneous with the unit and *G*, by the unit. The result must be the magnitude *D* itself, and not an object or quantity of a higher dimension.

Some historians have inferred from the context of the proof that al-Khayyām's "numbers" are conceived as being the measures of ratios of magnitudes.[43] But al-Khayyām defines *G* in the context of ratios only because this particular theorem is about ratios. As we saw in the algebraic works, his "numbers" derive instead from the abstraction of "quantity" from

[41] [Rashed & Vahabzadeh 1999, 123.11,19/2000, 114] (their translation).
[42] [Rashed & Vahabzadeh 1999, 379.11/2000, 253] (adapted from their translation).
[43] Two accounts of al-Khayyām's "numbers" are [Youschkevitch 1976, 87-88; Vahabzadeh 2004].

continuous magnitudes. While his numbers *can* be regarded as the measures of ratios, they are not ratios essentially.

8. Summary of al-Khayyām's algebra

Practical algebraists calculated in a number system comprising all quantities that might arise in calculation or measurement, including fractions and irrational roots. This system was well suited for the kind of geometrical problem solving practiced by al-Khayyām and others, but it needed a rigorous foundation. For this al-Khayyām turned to Aristotle's genus of "quantity". The species that make up this genus come in two types. For Aristotle "numbers" (positive integers) are an example of a discrete type, because the unit in arithmetic is indivisible. "Lines", "surfaces", "bodies", and "time" are continuous types, because the units in these species are divisible. Because the algebraists worked with a divisible unit, al-Khayyām identified their numbers with the measures of continuous magnitudes.

These "continuous numbers", as I call them, are not associated with any particular species of magnitude. A number like "two and a half" can be the length of a line, the area of a rectangle, the volume of a cube, or the length of a span of time. They embody the abstract quality of "quantity" shared by continuous magnitudes. Because the numbers are not substantiated in "material things" (i.e. in any species), they can only be "abstracted by the mind". One advantage of such an abstract system is that the numbers are closed under multiplication. Had al-Khayyām worked with geometric magnitudes themselves, there would be a problem with dimension. Multiplying two lines gives a plane, not another line, and one cannot multiply two plans or two solids together at all. But the product of two continuous numbers always yields another continuous number. This property is necessary for polynomial algebra and for al-Khayyām's proof in the third part of his *Commentary*.

Al-Khayyām works with the same unbounded range of powers as in practical algebra, and with the same names: *shay'* ("thing") or *jidhr* ("root") for the first power, *māl* ("sum of money") for the second, *ka'b* ("cube") for the third, followed by the higher powers *māl māl*, *māl ka'b*, *ka'b ka'b*, as high as one wishes to go. Near the end of his *Algebra* al-Khayyām even works with the reciprocals of the powers.[44] Being continuous numbers, the powers are homogeneous and proportional, so no power is tied to any dimension. In the course of solving a geometry problem, for instance, a *māl* can be the measure of a line, a plane, or a solid.

In his *Quadrant* al-Khayyām maintained that algebra can only be used to solve geometry problems. The unknowns, after all, are the abstraction of *continuous* measure. But later in his *Algebra* he wrote that algebra can be used to solve arithmetic questions, too.[45] The continuous numbers can be taken to represent positive integers as well as the measures of geometric magnitudes, though there would probably not be much demand for the former.

The structure of the solution to a problem in al-Khayyām's algebra differs from that in practical algebra only in stage 3:

> Stage 1: An unknown number, either discrete (arithmetic) or continuous (geometry), is named in terms of the algebraic powers (usually as a "thing"). The operations or conditions specified in the enunciation are then applied to arrive at an equation.
>
> Stage 2: The equation is simplified to a standard form. Al-Khayyām lists the twenty-five standard polynomial equations of the first three degrees in both treatises.
>
> Stage 3: If the enunciation asks for an unknown number, then one follows the prescribed numerical solution given in the *Algebra*. If the original unknown is a line, then one performs the prescribed geometric construction to produce a line whose length is the "thing".[46]

Al-Khayyām's *Algebra* is mainly a guide for performing stage 3, where the nature of the unknowns as abstract, dimensionless "quantity" is reinterpreted either as number or geometric magnitude. For equations of

[44] [Rashed & Vahabzadeh 1999, 217/2000, 156]. Al-Karajī worked with the reciprocals of the powers in his *al-Fakhrī* (early 11th c.).

[45] In addition to the three geometrical degrees, time is a kind of continuous magnitude, and al-Khayyām notes that algebra could also be used to solve problems in horology [Rashed & Vahabzadeh 1999, 121.18/2000, 113].

[46] Al-Khayyām did not give solutions for equations in which the unknown is the *māl* or the "cube".

degree one and two he gives two solutions. If the original problem is in arithmetic, one follows the arithmetical rule. If it is in geometry, one follows the geometric construction. Al-Khayyām gives geometric proofs for both kinds of solutions. He gives only geometric solutions and proofs for irreducible cubic equations because he was not able to solve them numerically.

The answer to an arithmetic problem must be a number, so al-Khayyām's rule produces the answer from the numbers (coefficients) of the terms in the simplified equation, just as al-Khwārizmī and other algebraists had done. Al-Khayyām's geometric proofs for the numerical rules follow the tradition established by al-Khwārizmī and Ibn Turk, and later improved by Abū Kāmil and al-Karajī.

The answer to a geometry problem must be given as a geometric magnitude. The geometry problems suitable for algebra are those that ask for a magnitude whose *measure* satisfies a certain condition. Typically this measure is named a "thing", and the resulting simplified equation is solved by constructing a line whose length is the value of the "thing". Al-Khayyām proves that the constructed line satisfies the equation by reinterpreting "thing", *māl*, and "cube" as the line, the square on the line, and the cube on the line respectively.

While al-Khayyām puts no limit on the degree of the equations in stages 1 and 2, he caps the degree of simplified equations to be solved in stage 3 at the third degree. This is due to the identification of the powers of the unknown with geometric magnitudes in the proofs of the geometric solutions, as shown in section §6.

9. Terminology

In reading over al-Khayyām's mathematical works one finds the word "number" used with different meanings in different contexts, and in the *Algebra* the names of the powers are sometimes confused with the names of their corresponding geometric magnitudes. It is necessary to distinguish between these meanings when reading his works, so I outline the potentially confusing terms here.

Al-Khayyām uses the word *ʻadad* ("number") four different ways. The context of each use of the term is enough to determine which meaning he has in mind:

1) As the discrete objects of arithmetic, "numbers" are positive integers.

2) In his *Commentary* he speaks of his notion of "number", which is "abstracted by the mind" as the dimensionless measure of continuous magnitudes. These continuous numbers form the foundation for al-Khayyām's algebra. The powers "thing", *māl*, etc. are regarded as unknown numbers in this sense, so the use of the word "number" for the zero degree term in a polynomial should be understood as taking the same meaning.

3) In the proof of the geometric solution to an equation "number" is reinterpreted as the measure of a solid (for cubic equations) or a plane (for quadratic equations).

4) Al-Khayyām follows earlier algebraists in speaking of the "number" of a term as *how many* there are. So the number of "things" in the polynomial "a *māl* and three things" is "three". In modern terms, this number is the coefficient of the term. In practical Arabic algebra and in al-Khayyām this number cannot be irrational, but it can be a fraction.[47]

In his *Quadrant* al-Khayyām distinguishes between the algebraic powers and their geometric counterparts by his choice of terms. During stages 1 and 2 he works with the traditional algebraic words *shay'* ("thing"), *māl*, *ka'b* ("cube"), and *māl māl*. He begins stage 3 by looking for a geometric *ḍil'* ("side"), *murabba'* ("square"), and *muka''b* ("*cube*"), distinguishing even between the algebraic and the geometric "cube". But in the *Algebra* al-Khayyām is not so careful with his choice of words. In his initial list of the 25 equations he uses the proper algebraic terms, but when he restates the equations before giving their solutions, he sometimes mixes the words. For the equations with numerical solutions (degree 1 and 2) he keeps the proper algebraic terms, but for the 14 irreducible cubic equations, which are solved

[47] For example, Abū Kāmil multiplies "the root of five" by "a thing" to get "the root of five *māl*s", i.e. $\sqrt{5}\cdot x \to \sqrt{5x^2}$ [Abū Kāmil 1986, 89.12]. The root is placed over the whole term to keep the coefficient rational. See [Oaks 2009], particularly §5.2.

only by geometry, the geometric terms *ḍil'* and *muka''b* replace *shay'* and *ka'b*.[48]

The intermingling of algebraic and geometric terms in the *Algebra* does not hold any significance. In the proofs the algebraic powers are to be identified with geometric magnitudes anyway, so al-Khayyām has no real motive to distinguish them. Also, he was lax in a few other places in the *Algebra*. For example, in most equations al-Khayyām states the general case, but for four types he states and solves specific equations: $x^2 = 5x$, $x^2 + 10x = 39$, $x^3 + 2x = 3x^2$, and $x^3 = x^2 + 3x$. The first two are the standard examples from the practical tradition. Also, he sometimes uses the phrase "the number of" a term to mean the whole term, and not just the coefficient. Last, in the geometric proof of the numerical solution to $x^2 = ax + b$ he works with the example $x^2 = 5x + 7$, whose solution is irrational. While the geometric solutions and proofs in al-Khayyām's *Algebra* are well conceived and presented, other aspects of his book are not so polished.

10. Influences

The influences of earlier algebraists on al-Khayyām are clear. His numerical solutions and proofs for linear and quadratic equations show elements taken from al-Khwārizmī's book, including one proof of the numerical solution to $x^2 + 10x = 39$. The idea of solving irreducible cubic equations with conic sections goes back to al-Khāzin, and al-Khayyām mentions that Abū'l-Jūd ibn al-Layth (10th-11th c.) had previously attempted to classify and solve all cubic equations.

But what about the concept of continuous number? The theoretical geometers in the two centuries before al-Khayyām who sought to appropriate algebra for problem solving would not have accepted the "numbers" of the practical algebraists. Matvievskaya has noted that al-Khāzin based his *Commentary on Book X of Euclid's Work* on Aristotle's

[48] He also uses the geometric "cube" in stating the reducible cubic equations.

classification of continuous magnitudes.[49] It is likely that al-Māhānī, al-Khāzin, Abū'l-Jūd, and other geometers who worked with cubic equations before al-Khayyām also regarded the algebraic powers as being the abstraction of "quantity". This would help to explain why al-Khayyām does not explicitly address the nature of his numbers. Later geometers continued to ground their notion of continuous number in the same way as al-Khayyām, including Qāḍī Zāda al-Rūmī in his commentary on al-Samarqandī's *Ashkāl al-ta'sīs* (1412).[50]

Al-Khayyām's only successor in algebra whose work survives is Sharaf al-Dīn al-Ṭūsī (d. 1213/14). Sharaf al-Dīn wrote his *Problems of Algebra* (*Masā'il al-jabr wa'l-muqābala*) with the same general aim as al-Khayyām: to present solutions to the twenty-five simplified equations of degree three and less, with proofs. Sharaf al-Dīn does not discuss the nature of his unknowns, so we can presume that he upheld al-Khayyām's interpretation.

Sharaf al-Dīn included numerical solutions to *all* equations. These solutions are given in terms of specific equations, solved by what is now known as the Ruffini-Horner method. He uses root extraction even for quadratic equations, thus bypassing the traditional solutions dating back to al-Khwārizmī. Because numbers are restricted to integers, this process always yields the exact answer in a finite number of steps. Al-Khayyām mentioned the numerical method of root extraction in his treatment of the equations $x^2 = a$ and $x^3 = a$, but he did not include it in his *Algebra* because his own proofs were not grounded in geometry: "But these demonstrations are only numerical demonstrations based on the arithmetical Books of the *Elements*."[51] Another modification in Sharaf al-Dīn is that in addition to lines, his unknowns can be the measures of two- and three-dimensional geometric objects.

Al-Khayyām's contributions to algebra had no noticeable impact on practical Arabic algebra, and by extension, the algebra imported by medieval Europeans in Latin and Italian.[52] His grounding of "number" in abstract continuous "quantity" is a philosophical nicety that merely justifies earlier

[49] [Matvievskaya 1987, 261; Brentjes 2008, 448]. We also find Abū'l-Jūd giving numerical values to lines and areas, and calling a particular *area* a "fourth binomial" in his construction of the regular heptagon [Hogendijk 1987].
[50] [Fazlioğlu 2008, 27].
[51] [Rashed & Vahabzadeh 1999, 131.2/2000, 117] (their translation).
[52] The only whisper of influence is the presence of al-Khayyām's equation $x^3 + 2x^2 + 10x = 20$ in Fibonacci's *Flos*. [Rashed 1994, 149].

practice, and his geometric solutions to cubic equations would have been no use to an architect or accountant seeking a numerical solution. Even those practical algebraists who showed a more theoretical interest in algebra and who worked with cubic equations, such as Ibn al-Hāʼim (1387) and Ibn al-Majdī (d. 1447), do not mention al-Khayyām or Sharaf al-Dīn al-Ṭūsī.[53]

Even theoretical mathematicians working in geometry and astronomy make few references to our algebraist. Ahmed Djebbar summed it up with "En effet les informations disponibles actuellement, sur un éventuel enseignement de Khayyām et sur la circulation directe ou indirecte de ses écrits mathématiques sont rares et souvent vagues et imprécises."[54]

11. Conclusion

The interplay between practical and pure mathematics in medieval Islam is vividly evident in the field of algebra. Al-Karajī, for example, is not unusual in writing books for practitioners (his *al-Kāfī*) and also for theoretical mathematicians (the *Badīʻ*). His major text on algebra, *al-Fakhrī*, combines practical instruction with original investigations into the arithmetic of polynomials.

Al-Khayyām's *Algebra* gives us a view of another way practice met theory. This book is a guide to algebra for use by geometers working in the Greek tradition. Where al-Karajī and others brought the rigor of theoretical mathematics to practical algebra, al-Khayyām was among those who made practical algebra available to theoretical geometers. Because the liberal notion of number which formed the basis of practical algebra was at odds with the strict distinction made by geometers between discrete numbers and continuous magnitudes, al-Khayyām, and surely also those who worked before him, regarded the algebraic unknowns not as numbers, but as the dimension-free abstraction of continuous "quantity".

Algebra for al-Khayyām remained a method of solving problems in geometry and arithmetic. From the perspective of the mathematical objects involved, the algebraic solution begins by representing the unknown magnitude or positive integer by the abstract, dimensionless "thing". After setting up and simplifying the equation to one of the twenty-five standard forms, the solution to the original problem is found by reinterpreting the

[53] [Ibn al-Hāʼim 2003; Djebbar 2000, 26-31].
[54] [Djebbar 2000, 26].

"thing" as a magnitude or number. (The constructions in al-Khayyām's *Algebra* belong only to this last step.)

Looking at it from the perspective of problems and equations, a geometry or arithmetic problem is translated into an algebraic equation, which is then simplified. This simplified equation is then reinterpreted as a new problem to be solved. It is the simplification of the equation, which for al-Khayyām takes place in the domain of abstract, dimensionless quantities, which allows the algebraist to transform a given problem into a new one whose solution is provided in his book.

Sources

Abū Kāmil, 1986. *Kitāb fī al-Jabr wa'l-Muqābala*. A facsimile edition of MS Istanbul, Kara Mustafa Paşa 379, copied in 1253 C.E. Edited by Jan P. Hogendijk. Frankfurt am Main: Institute for the History of Arabic-Islamic Science at the Johann Wolfgang Goethe University.

Aristotle, 1963. *Aristotle's* Categories *and* De Interpretatione. Translated with Notes by J. L. Ackrill. Oxford: Oxford University Press, 1963.

Brentjes, Sonja, 2008. "Euclid's *Elements*, courtly patronage, and princely education." *Iranian Studies* 41, pp. 441-463.

Cuomo, S., 2000. *Pappus of Alexandria and the Mathematics of Late Antiquity*. Cambridge: Cambridge University.

Djebbar, Ahmed, 2000. " 'Omar Khayyām et les Activités Mathématiques en Pays d'Islam aux XI^e^-XII^e^ Siècles." *Farhang* 12.29-32, 1-31.

Fazlioğlu, İhsan, 2008. "The Samarqand mathematical-astronomical school: A basis for Ottoman philosophy and science." *Journal for the History of Arabic Science* 14, 3-68.

Gutas, Dimitri, 1998. *Greek Thought, Arabic Culture: The Graeco-Arabic Translation Movement in Baghdad and Early 'Abbāsid Society (2nd-4th/8th-10th centuries)*. London: Routledge.

Hogendijk, Jan P., 1987. "Abu'l-Jūd's answer to a question of al-Bīrūnī concerning the regular heptagon." Pp. 175-183 in: David A. King and

George Saliba, edd., *From Deferent to Equant: A Volume of Studies in the History of the Exact Sciences in the Ancient and Medieval Near East in Honor of E. S. Kennedy*. New York: New York Academy of Sciences.

Ibn al-Hā'im, 2003. *Sharḥ al-Urjūza al-Yāsmīnīya, de Ibn al-Hā'im*. Texte établi et commenté par Mahdi Abdeljaouad. Tunis: Publication de l'Association Tunisienne des Sciences Mathématiques.

Al-Khwārizmī, 1831. *The Algebra of Mohammed ben Musa*, translated and edited by Frederic Rosen. London: Nachdruck der Ausgabe. (Reprinted 1997 by the Institute for the History of Arabic-Islamic Science.)

Al-Khwārizmī, 2007. *Al-Khwārizmī: Le Commencement de l'Algèbre*. Texte établi, traduit et commenté par Roshdi Rashed. Paris: Blanchard.

Matvievskaya, Galina, 1987. "The theory of quadratic irrationals in medieval oriental mathematics." Pp. 253-277 in: David A. King and George Saliba, edd., *From Deferent to Equant: A Volume of Studies in the History of the Exact Sciences in the Ancient and Medieval Near East in Honor of E. S. Kennedy*. New York: New York Academy of Sciences.

Netz, Reviel, 2004. *The Works of Archimedes. Volume I: The Two Books* On the Sphere and the Cylinder. Translated into English, together with Eutocius' commentaries, with commentary and critical edition of the diagrams. Cambridge: Cambridge University Press.

Oaks, Jeffrey A. and Haitham M. Alkhateeb, 2007. "Simplifying Equations in Arabic Algebra". *Historia Mathematica* 34, 45-61.

Oaks, Jeffrey A., 2009. "Polynomials and equations in Arabic algebra". *Archive for History of Exact Sciences* 63, 169-203.

Panza, Marco, 2007. "Relations between algebra and analysis before Viète". *Revue d'Histoire des Mathématiques* 13, 83-151.

Rashed, Roshdi, and Ahmad Djebbar, 1981. *L'Oeuvre Algébrique d'al-Khayyām*. Établie, traduite et analysée par Roshdi Rashed et Ahmad Djebbar. Aleppo: Institute for the History of Arabic Science.

Rashed, Roshdi, 1994. "Fibonacci et les mathématiques arabes". Pp. 145-160 in: Michael McVaugh and Véronique Pasche, eds. *Sciences at the court of Frederick II*. Turnhout: Brepols.

Rashed, Roshdi, 1996. *Les mathématiques infinitésimales du IXe au XIe siècle, Fondateurs et commentateurs. Volume I: Banū Mūsā, Ibn Qurra, Ibn Sinān, al-Khāzin, al-Qūhī, Ibn al-Samḥ, Ibn Hūd*. London: Al-Furqan Islamic Heritage Foundation.

Rashed, Roshdi, and Bijan Vahabzadeh, 1999. *Al-Khayyām Mathématicien*. Paris: Blanchard.

Rashed, Roshdi, and Bijan Vahabzadeh, 2000. *Omar Khayyam, the Mathematician*. New York, Winona Lake, Ind.: Bibliotheca Persica Press. (English translation of [Rashed & Vahabzadeh 1999].)

Rosenfeld, Boris A. and Ihsanoğlu, Ekmeleddin, 2003. *Mathematicians, Astronomers, and Other Scholars of Islamic Civilisation and their works (7th-19th c.)*. Istanbul: Research Centre for Islamic History Art and Culture (IRCICA).

Sharaf al-Dīn al-Ṭūsī, 1986. *Œuvres mathématiques: Algèbre et géométrie au XIIe siècle*. Two vols. Texte établi et traduit par Roshdi Rashed. Paris: Les Belles Lettres.

Vahabzadeh, Bijan, 2004. " 'Umar al-Khayyām and the concept of irrational number." Pp. 55-63 in: Regis Morelon and Ahmad, edd., *De Zénon d'Elée à Poincaré: recueil d'études en hommage à Roshdi Rashed*. Louvain: Peeters.

Woepcke, Franz, 1851. *L'algèbre d'Omar Alkhayyāmī*, publiée, traduite et accompagnée d'extracts de manuscrits inédits. Paris. (Reprinted 1998 by Institute for the History of Arabic-Islamic Science at the Johann Wolfgang Goethe University.)

Youschkevitch, Adolf P., 1976. *Les Mathématiques Arabes (VIII^e^-VX^e^ siècles)*. Traduction par M. Cazenave et K. Jaouiche, Préface de René Taton. Paris: Vrin.